COORDINATE SOLID GEOMETRY

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PUBLISHER'S NOTE

PROFESSOR R. J. T. BELL'S *Elementary Treatise on Coordinate Geometry of Three Dimensions* embodied a course of geometry which was designed to meet the needs of students of both pure and applied mathematics. For the former it is an introduction to longer works on the subject which they will come to later in their course; for the latter, it gives a fairly complete exposition of the properties of the plane, the straight line and the conicoids.

It is realised, however, that there is now a number of students, whose interest in pure mathematics will not extend beyond the requirements of pass degree examinations in mathematics. For them Chapters I-IX of the complete treatise, completing the work on the central surfaces, is sufficient. It has accordingly been arranged to issue the first nine chapters of the book as a separate book. For this purpose the author has selected the appropriate Miscellaneous Examples (II) and has provided a new set in addition (I). He has also added an appendix suggesting some simplified methods he has developed since the original book was written, and made a number of minor alterations.

May, 1938.

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CHAPTER I.

SYSTEMS OF COORDINATES. THE EQUATION TO A SURFACE.

1. Segments. Two segments AB and CD are said to have the same direction when they are collinear or parallel, and when B is on the same side of A as D is of C. If AB and CD have the same direction, BA and CD have opposite directions. If AB and CD are of the same length and in the same direction they are said to be **equivalent segments.**

2. If A, B, C, ... N, P are any points on a straight line X′OX, and the convention is made that a segment of the straight line is positive or negative according as its direction is that of OX or OX′, then we have the following relations:

$$\mathrm{AB}=-\mathrm{BA}\,;\quad \mathrm{OA}+\mathrm{AB}=\mathrm{OB},\quad \text{or}\quad \mathrm{AB}=\mathrm{OB}-\mathrm{OA},$$

$$\text{or}\quad \mathrm{OA}+\mathrm{AB}+\mathrm{BO}=0\,;$$

$$\mathrm{OA}+\mathrm{AB}+\mathrm{BC}+\ldots\,\mathrm{NP}=\mathrm{OP}.$$

If x_1, x_2 are the measures of OA and OB, *i.e.* the ratios of OA and OB to any positive segment of unit length, L, then

$$\mathrm{OA}=x_1\mathrm{L},\quad \mathrm{OB}=x_2\mathrm{L},$$

and

$$\mathrm{AB}=(x_2-x_1)\,\mathrm{L},$$

or the measure of AB is x_2-x_1.

3. Coordinates. Let X′OX, Y′OY, Z′OZ be any three fixed intersecting lines which are not coplanar, and whose positive directions are chosen to be X′OX, Y′OY, Z′OZ; and let planes through any point in space, P, parallel respectively to the planes YOZ, ZOX, XOY, cut X′X, Y′Y, Z′Z in A, B, C, (fig. 1), then the position of P is known when the segments

OA, OB, OC are given in magnitude and sign. A construction for P would be: cut off from OX the segment OA, draw AN, through A, equivalent to the segment OB, and draw NP, through N, equivalent to the segment OC. OA, OB, OC are known when their measures are known, and these measures are called the **Cartesian coordinates** of P with reference to the coordinate axes X′OX, Y′OY, Z′OZ. The point O is called the origin and the planes YOZ, ZOX, XOY, the coordinate planes. The measure of OA, the segment cut off from OX or OX′ by the plane through P parallel to YOZ, is called the x-coordinate of P; the measures of OB and OC are the y and z-coordinates, and the symbol P, (x, y, z) is used to denote, "the point P whose coordinates are x, y, z." The coordinate planes divide space into eight parts called octants, and the signs of the coordinates of a point determine the octant in which it lies. The following table shews the signs for the eight octants:

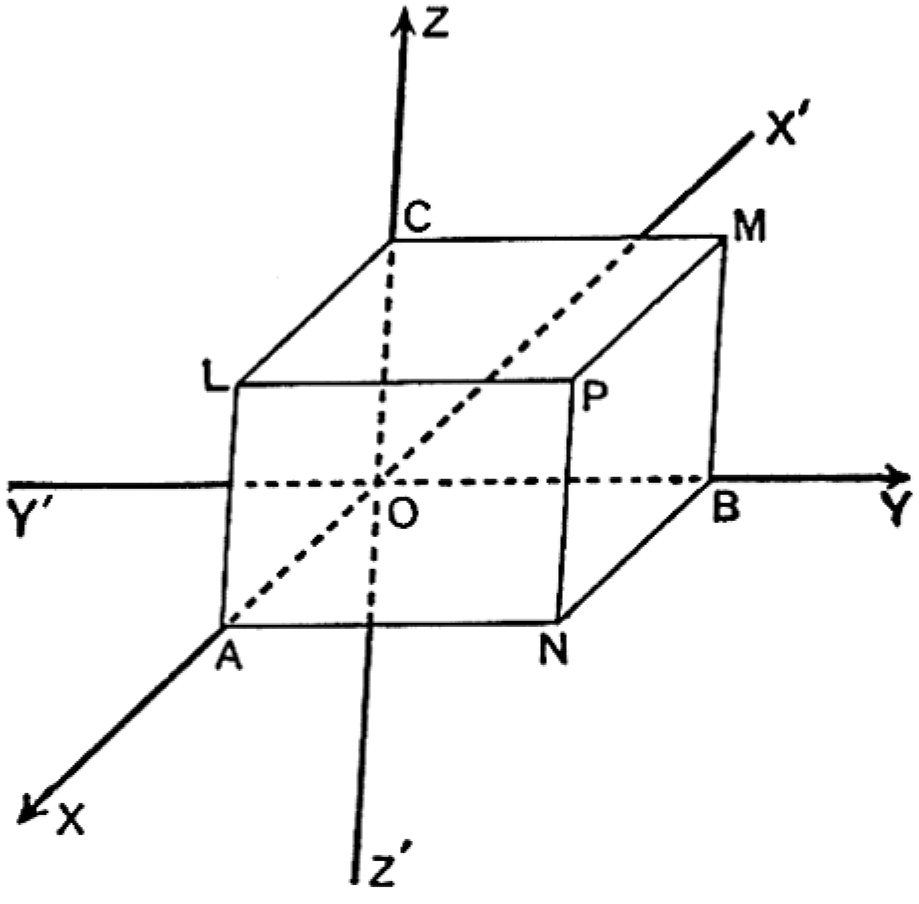

Fig. 1.

Octant	OXYZ	OX′YZ	OX′Y′Z	OXY′Z	OXYZ′	OX′YZ′	OX′Y′Z′	OXY′Z′
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

It is generally most convenient to choose mutually perpendicular lines as coordinate axes. The axes are then "rectangular," otherwise they are "oblique."

Ex. 1. Sketch in a figure the positions of the points:

$$(8, 0, 3),\ (-2, -1, 5),\ (-4, -2, 0),\ (0, 0, -6).$$

Ex. 2. What is the locus of the point, (i) whose x-coordinate is 3, (ii) whose x-coordinate is 2 and whose y-coordinate is -4?

Ex. 3. What is the locus of a point whose coordinates satisfy (i) $x=0$ and $y=0$; (ii) $x=a$ and $y=0$; (iii) $x=a$ and $y=b$; (iv) $z=c$ and $y=b$?

Ex. 4. If OA $=a$, OB $=b$, OC $=c$, (fig. 1), what are the equations to the planes PNBM, PMCL, PNAL? What equations are satisfied by the coordinates of any point on the line PN?

4. Sign of direction of rotation. By assigning positive directions to a system of rectangular axes X′X, Y′Y, Z′Z, we have fixed the positive directions of the normals to the coordinate planes YOZ, ZOX, XOY. Retaining the usual convention made in plane geometry, the positive direction of rotation for a ray revolving about O in the plane XOY is that given by XYX′Y′, that is, is counter-clockwise, if the clock dial be supposed to coincide with the plane and front in the positive direction of the normal. Hence to fix the positive direction of rotation for a ray in *any* plane, we have the rule: *if a clock dial is considered to coincide with the plane and front in the positive direction of the normal to the plane, the positive direction of rotation for a ray revolving in the plane is counter-clockwise.* Applying this rule to the other coordinate planes the positive directions of rotation for the planes YOZ, ZOX are seen to be YZY′Z′, ZXZ′X′.

The positive direction of rotation for a plane can also be found by considering that it is the direction in which a right-handed gimlet or corkscrew has to be turned so that it may move forward in the positive direction of the normal to the plane.

Ex. A plane ABC meets the axes OX, OY, OZ in A, B, C, and ON is the normal from O. If ON is chosen as the positive direction of the normal, and a point P moves round the perimeter of the triangle ABC in the direction ABC, what is the sign of the direction of rotation of NP when OA, OB, OC are (i) all positive, (ii) one negative, (iii) two negative, (iv) all negative?

5. Cylindrical coordinates. If X′OX, Y′OY, Z′OZ, are rectangular axes, and PN is the perpendicular from any point P to the plane XOY, the position of P is determined if ON, the angle XON, and NP are known. The measures of these quantities, u, ϕ, z, are the **cylindrical coordinates** of P. The positive direction of rotation for the plane XOY has been defined, and the direction of a ray originally coincident with OX, and then turned through the given angle ϕ, is the positive direction of ON. In the figure, u, ϕ, z are all positive.

If the Cartesian coordinates of P are x, y, z, those of N are x, y, 0. If we consider only points in the plane XOY, the Cartesian coordinates of N are x, y, and the polar, u, ϕ Therefore

$$x = u\cos\phi,\ y = u\sin\phi;\quad u^2 = x^2 + y^2,\ \tan\phi = y/x.$$

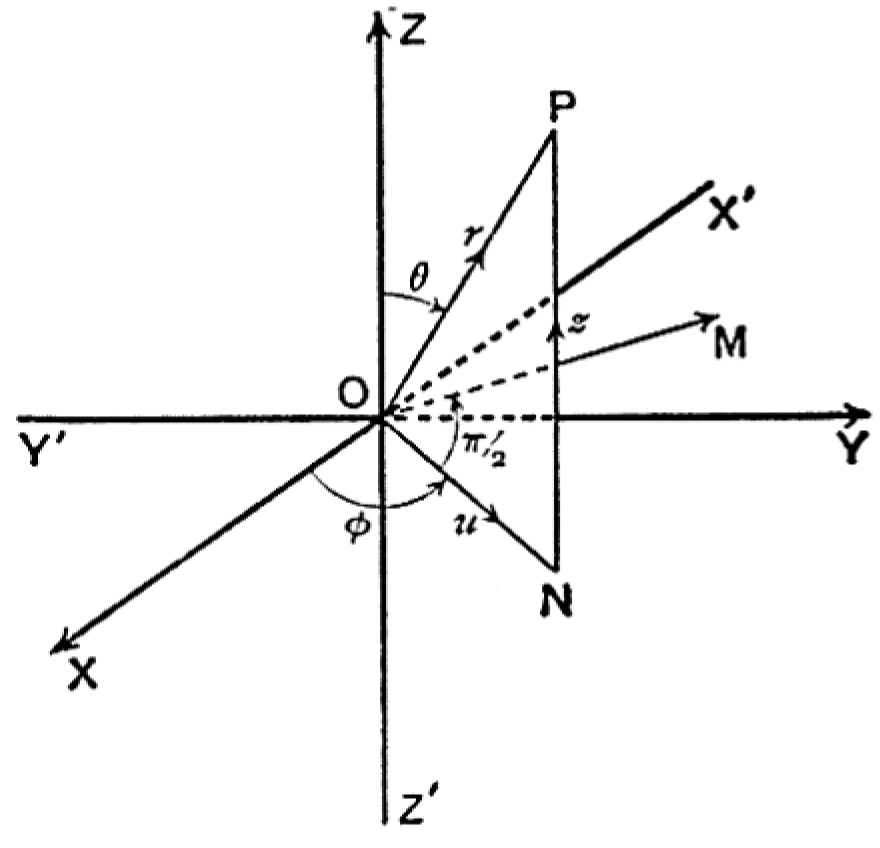

FIG. 2.

6. Polar coordinates. Suppose that the position of the plane OZPN, (fig. 2), has been determined by a given value of ϕ, then we may define the positive direction of the normal through O to the plane to be that which makes an angle $\phi + \pi/2$ with X′OX. Our convention, (§ 4), then fixes the positive direction of rotation for a ray revolving in the plane OZPN. The position of P is evidently determined when, in addition to ϕ, we are given r and θ, the measures of OP and ∠ZOP. The quantities r, θ, ϕ are the **polar coordinates** of P. The positive direction of OP is that of a ray originally coincident with OZ and then turned in the plane OZPN through the given angle θ. In the figure, OM is the positive direction of the normal to the plane OZPN, and r, θ, ϕ are all positive.

If we consider P as belonging to the plane OZPN and OZ and ON as rectangular axes in that plane, P has Cartesian coordinates z, u, and polar coordinates r, θ. Therefore

$$z=r\cos\theta,\quad u=r\sin\theta;\quad r^2=z^2+u^2,\ \tan\theta=\frac{u}{z}.$$

But if P is (x, y, z), $x=u\cos\phi$, $y=u\sin\phi$.

Whence $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$;

$$r^2=x^2+y^2+z^2,\ \tan\theta=\frac{\pm\sqrt{x^2+y^2}}{z},\ \tan\phi=\frac{y}{x}.$$

Cor. If the axes are rectangular the distance of (x, y, z) from the origin is given by $\sqrt{x^2+y^2+z^2}$.

Ex. 1. Draw figures shewing the positions of the points

$$\left(2, -\frac{\pi}{3}, \frac{\pi}{4}\right),\ \left(3, \frac{2\pi}{3}, -\frac{\pi}{6}\right),\ \left(-3, -\frac{\pi}{4}, -\frac{\pi}{3}\right),\ \left(2, \frac{5\pi}{6}, -\frac{3\pi}{4}\right).$$

What are the Cartesian coordinates of the points?

Ex. 2. Find the polar coordinates of the points $(3, 4, 5)$, $(-2, 1, -2)$, so that r may be positive.

Ans. $\left(5\sqrt{2}, \frac{\pi}{4}, \tan^{-1}\frac{4}{3}\right)$, $\left(3, \frac{\pi}{2}+\tan^{-1}\frac{2\sqrt{5}}{5}, \frac{\pi}{2}+\tan^{-1}2\right)$,

where $\tan^{-1}\frac{4}{3}$, $\tan^{-1}\frac{2\sqrt{5}}{5}$, $\tan^{-1}2$ are acute angles.

Ex. 3. Shew that the distances of the point $(1, 2, 3)$ from the coordinate axes are $\sqrt{13}$, $\sqrt{10}$, $\sqrt{5}$.

Ex. 4. Find (i) the Cartesian, (ii) the cylindrical, (iii) the polar equation of the sphere whose centre is the origin and radius 4.

Ans. (i) $x^2+y^2+z^2=16$, (ii) $u^2+z^2=16$, (iii) $r=4$.

Ex. 5. Find (i) the polar, (ii) the cylindrical, (iii) the Cartesian equation of the right circular cone whose vertex is O, axis OZ, and semivertical angle a.

Ans. (i) $\theta=a$, (ii) $u=z\tan a$, (iii) $x^2+y^2=z^2\tan^2 a$.

Ex. 6. Find (i) the cylindrical, (ii) the Cartesian, (iii) the polar equation of the right circular cylinder whose axis is OZ and radius a.

Ans. (i) $u=a$, (ii) $x^2+y^2=a^2$, (iii) $r\sin\theta=a$.

Ex. 7. Find (i) the polar, (ii) the Cartesian equation to the plane through OZ which makes an angle α with the plane ZOX.

Ans. (i) $\phi=\alpha$, (ii) $y=x\tan\alpha$.

7. Change of origin. Let X′OX, Y′OY, Z′OZ; $\alpha'\omega\alpha, \beta'\omega\beta, \gamma'\omega\gamma$, (fig. 3), be two sets of parallel axes, and let any point P be (x, y, z) referred to the first and (ξ, η, ζ) referred to the second set. Let ω have coordinates a, b, c, referred to OX, OY, OZ. NM is the line of intersection of the planes $\beta\omega\gamma$, XOY, and the plane through P parallel to $\beta\omega\gamma$ cuts $\alpha\omega\beta$ in GH and XOY in KL.

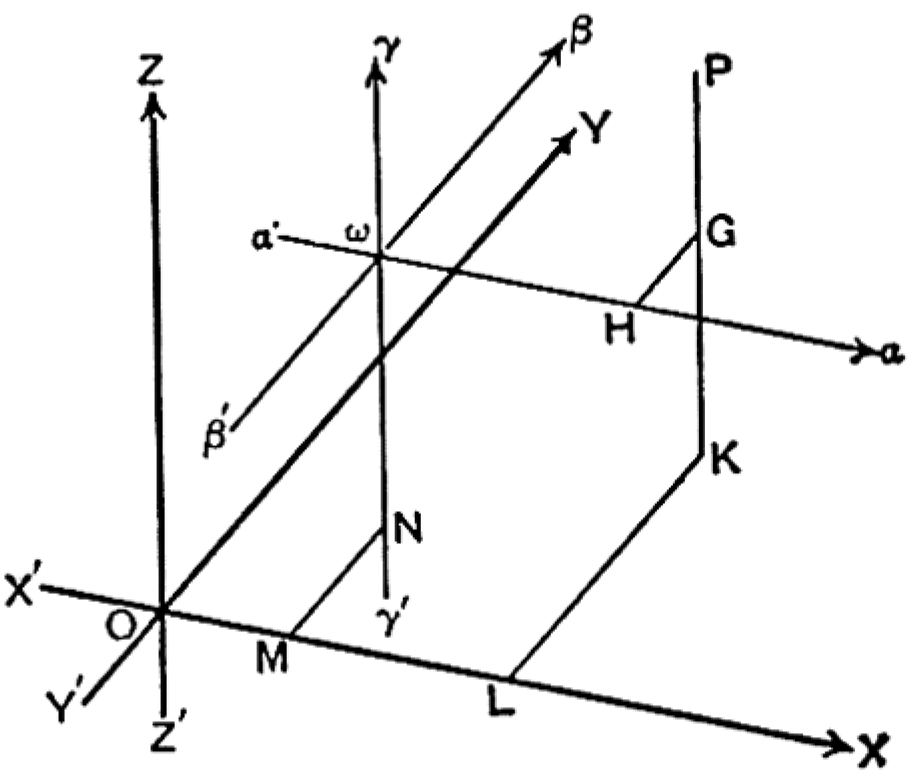

FIG. 3.

Then $\quad \text{OL}=\text{OM}+\text{ML}=\text{OM}+\omega\text{H},$

therefore $\quad x=a+\xi.\quad$ Similarly, $\quad y=b+\eta,\ z=c+\zeta;$

whence $\quad \xi=x-a,\ \eta=y-b,\ \zeta=z-c.$

Ex. 1. The coordinates of $(3, 4, 5)$, $(-1-5, 0)$, referred to parallel axes through $(-2, -3, -7)$, are $(5, 7, 12)$, $(1, -2, 7)$.

Ex. 2. Find the distance between P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2), the axes being rectangular.

Change the origin to P, and the coordinates of Q become x_2-x_1, y_2-y_1, z_2-z_1; and the distance is given by

$$\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}^{\frac{1}{2}}.$$

Ex. 3. The axes are rectangular and A, B are the points $(3, 4, 5)$, $(-1, 3, -7)$. A variable point P has coordinates x, y, z. Find the equations satisfied by x, y, z, if (i) PA=PB, (ii) $\text{PA}^2+\text{PB}^2=2k^2$, (iii) $\text{PA}^2-\text{PB}^2=2k^2$.

Ans. (i) $8x+2y+24z+9=0$,

(ii) $2x^2+2y^2+2z^2-4x-14y+4z+109=2k^2$,

(iii) $8x+2y+24z+9+2k^2=0$.

Ex. 4. Find the centre of the sphere through the four points (0, 0, 0), (0, 2, 0), (1, 0, 0), (0, 0, 4). *Ans.* $(\frac{1}{2}, 1, 2)$.

Ex. 5. Find the equation to the sphere whose centre is (0, 1, −1) and radius 2. *Ans.* $x^2+y^2+z^2-2y+2z=2$.

Ex. 6. Prove that $x^2-y^2+z^2-4x+2y+6z+12=0$ represents a right circular cone whose vertex is the point (2, 1, −3), whose axis is parallel to OY and whose semivertical angle is 45°.

Ex. 7. Prove that $x^2+y^2+z^2-2x+4y-6z-2=0$ represents a sphere whose centre is at (1, −2, 3) and radius 4.

8. *To find the coordinates of the point which divides the join of* P, (x_1, y_1, z_1) *and* Q, (x_2, y_2, z_2) *in a given ratio,* $\lambda:1$.

Let R, (x, y, z), (fig. 4), be the point, and let planes through P, Q, R, parallel to the plane YOZ, meet OX in P′, Q′, R′.

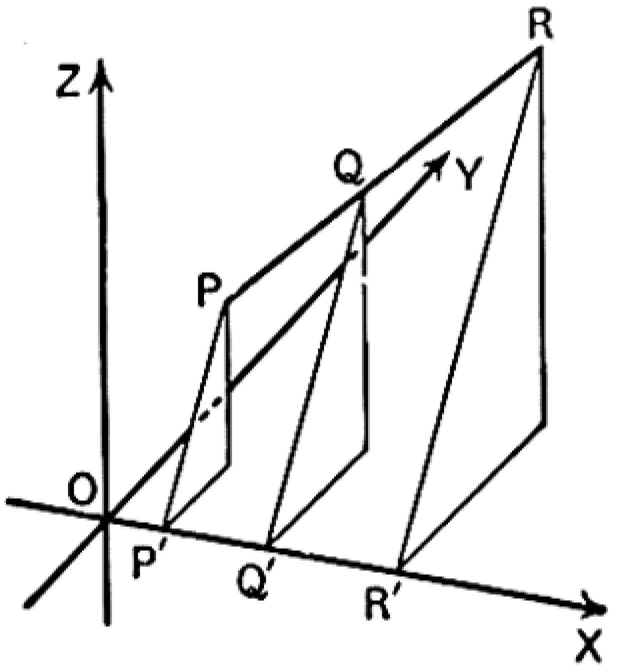

FIG. 4.

Then, since three parallel planes divide any two straight lines proportionally, $P'R' : P'Q' = PR : PQ = \lambda : \lambda+1$. Therefore

$$\frac{x-x_1}{x_2-x_1}=\frac{\lambda}{\lambda+1}, \quad \text{and} \quad x=\frac{\lambda x_2+x_1}{\lambda+1}.$$

Similarly,
$$y=\frac{\lambda y_2+y_1}{\lambda+1}, \quad z=\frac{\lambda z_2+z_1}{\lambda+1}.$$

These give the coordinates of R for all real values of λ positive or negative. If λ is positive, R lies between P and Q; if negative, R is on the same side of both P and Q.

Cor. The mid-point of PQ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Ex. 1. Find the coordinates of the points that divide the join of $(2, -3, 1)$, $(3, 4, -5)$ in the ratios $1:3$, $-1:3$, $3:-2$.

Ans. $\left(\frac{9}{4}, -\frac{5}{4}, -\frac{1}{2}\right)$, $\left(\frac{3}{2}, -\frac{13}{2}, 4\right)$, $(5, 18, -17)$.

Ex. 2. Given that P, $(3, 2, -4)$; Q, $(5, 4, -6)$; R, $(9, 8, -10)$ are collinear, find the ratio in which Q divides PR. Why can the ratio be found by considering the x-coordinates only? *Ans.* $1:2$.

Ex. 3. A, (x_1, y_1, z_1); B, (x_2, y_2, z_2); C, (x_3, y_3, z_3); D, (x_4, y_4, z_4) are the vertices of a tetrahedron. Prove that A′, the centroid of the triangle BCD, has coordinates

$$\frac{x_2+x_3+x_4}{3},\ \frac{y_2+y_3+y_4}{3},\ \frac{z_2+z_3+z_4}{3}.$$

If B′, C′, D′ are the centroids of the triangles CDA, DAB, ABC, prove that AA′, BB′, CC′, DD′ divide one another in the ratio $3:1$.

Ex. 4. Shew that the lines joining the mid-points of opposite edges of a tetrahedron bisect one another, and that if they be taken for coordinate axes, the coordinates of the vertices can be written (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$.

Ex. 5. Shew that the coordinates of any three points can be put in the form $(a, b, 0)$, $(a, 0, c)$, $(0, b, c)$, a fourth given point being taken as origin.

Ex. 6. The centres of gravity of the tetrahedra ABCD, A′B′C′D′, (Ex. 3), coincide.

Ex. 7. Find the ratios in which the coordinate planes divide the line joining the points $(-2, 4, 7)$, $(3, -5, 8)$. *Ans.* $2:3$, $4:5$, $-7:8$.

Ex. 8. Find the ratios in which the sphere $x^2+y^2+z^2=504$ divides the line joining the points $(12, -4, 8)$, $(27, -9, 18)$. *Ans.* $2:3$, $-2:3$.

Ex. 9. The sphere $x^2+y^2+z^2-2x+6y+14z+3=0$ meets the line joining A, $(2, -1, -4)$; B, $(5, 5, 5)$ in the points P and Q. Prove that $AP:PB=-AQ:QB=1:2$.

Ex. 10. A is the point $(-2, 2, 3)$ and B the point $(13, -3, 13)$. A point P moves so that $3PA=2PB$. Prove that the locus of P is the sphere given by

$$x^2+y^2+z^2+28x-12y+10z-247=0,$$

and verify that this sphere divides AB internally and externally in the ratio $2:3$.

Ex. 11. From the point $(1, -2, 3)$ lines are drawn to meet the sphere $x^2+y^2+z^2=4$, and they are divided in the ratio $2:3$. Prove that the points of section lie on the sphere

$$5x^2+5y^2+5z^2-6x+12y-18z+22=0.$$

9. The equation to a surface. *Any equation involving one or more of the current coordinates of a variable point represents a surface or system of surfaces which is the locus of the variable point.*

The locus of all points whose x-coordinates are equal to a constant α, is a plane parallel to the plane YOZ, and the equation $x=\alpha$ represents that plane. If the equation $f(x)=0$ has roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, it is equivalent to the equations $x=\alpha_1, x=\alpha_2, \ldots x=\alpha_n$, and therefore represents a system of planes, real or imaginary, parallel to the plane YOZ.

Similarly, $f(y)=0$, $f(z)=0$ represent systems of planes parallel to ZOX, XOY. In the same way, if polar coordinates be taken, $f(r)=0$ represents a system of spheres with a common centre at the origin, $f(\theta)=0$, a system of coaxal right circular cones whose axis is OZ, $f(\phi)=0$, a system of planes passing through OZ.

Consider now the equation $f(x, y)=0$. This equation is satisfied by the coordinates of all points of the curve in the plane XOY whose two-dimensional equation is $f(x, y)=0$

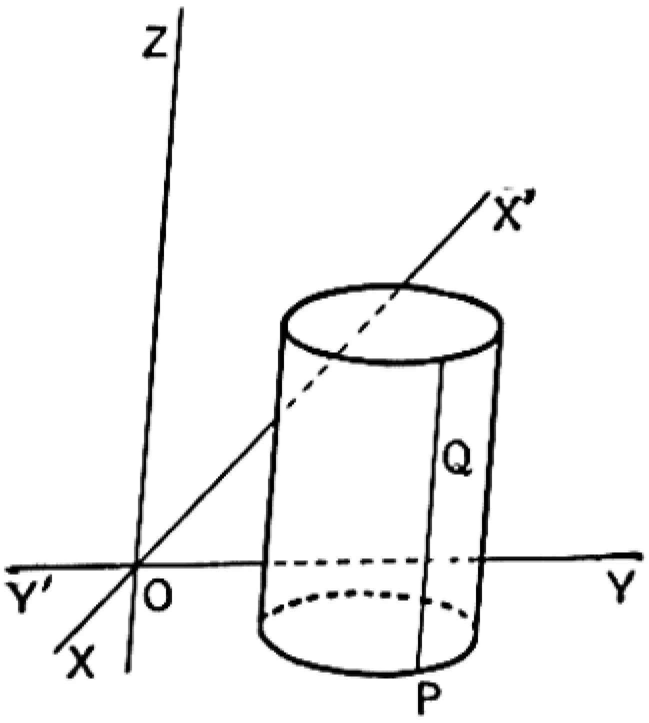

FIG. 5.

Let P, (fig. 5), any point of the curve, have coordinates $x_0, y_0, 0$. Draw through P a parallel to OZ, and let Q be any point on it. Then the coordinates of Q are x_0, y_0, z_0, and since P is on the curve, $f(x_0, y_0)=0$, thus the coordinates of Q satisfy the equation $f(x, y)=0$. Therefore the coordinates of every point on PQ satisfy the equation and every point on PQ lies on the locus of the equation. But P is any point of the curve, therefore the locus of the equation is the cylinder generated by straight lines drawn parallel to **OZ** through points of the curve. Similarly, $f(y, z)=0$, $f(z, x)=0$ represent cylinders generated by parallels to **OX** and **OY** respectively.

Ex. What surfaces are represented by (i) $x^2+y^2=a^2$, (ii) $y^2=4ax$, the axes being rectangular?

Two equations are necessary to determine the curve in the plane **XOY**. The curve is on the cylinder whose equation is $f(x, y)=0$ and on the plane whose equation is $z=0$, and hence "the equations to the curve" are $f(x, y)=0$, $z=0$.

Ex. What curves are represented by

(i) $x^2+y^2=a^2$, $z=0$; (ii) $x^2+y^2=a^2$, $z=b$; (iii) $z^2=4ax$, $y=c$?

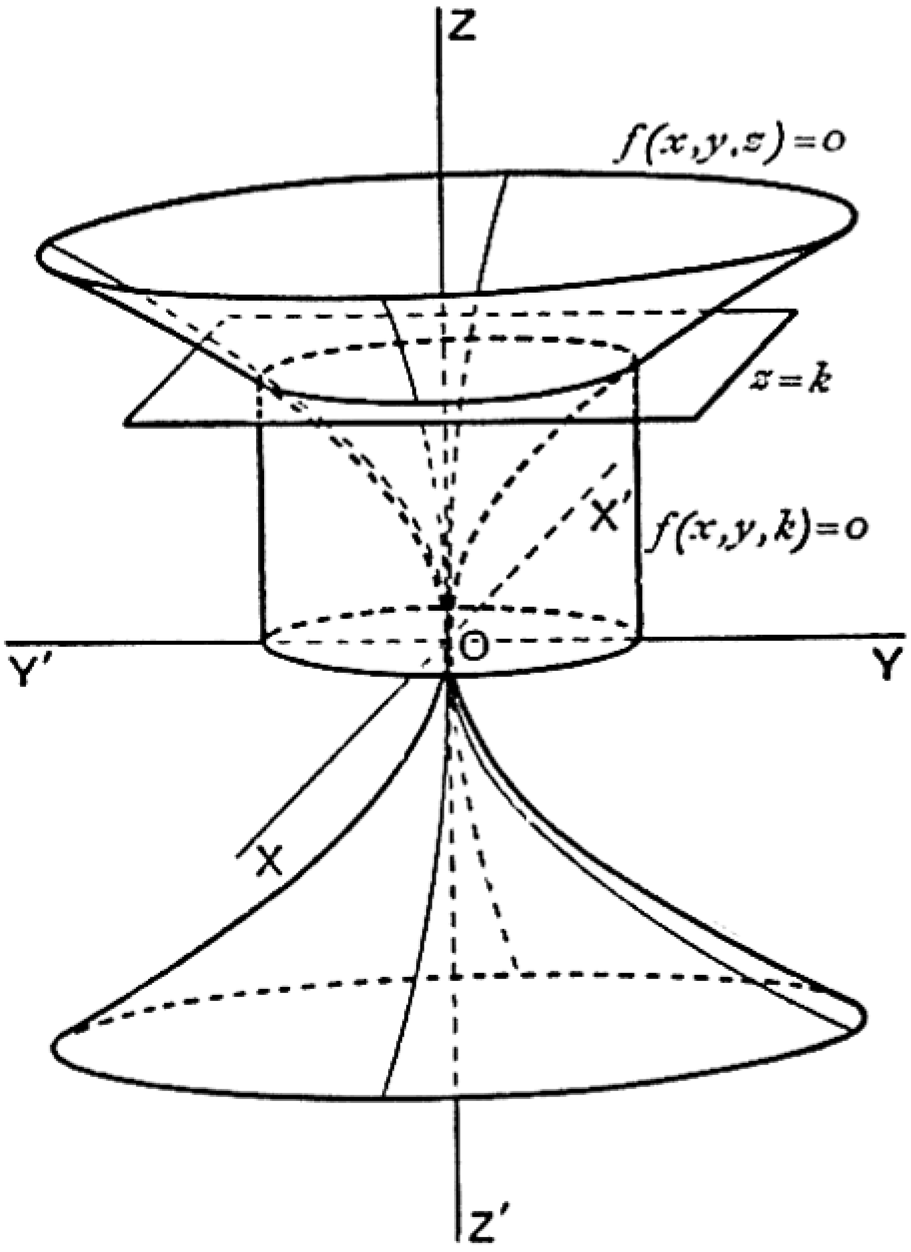

FIG. 6.

(The surface shewn is represented by the equation $a^2x^2+b^2y^2=z^4$.)

Consider now the equation $f(x, y, z)=0$. The equation $z=k$ represents a plane parallel to **XOY**, and the equation $f(x, y, k)=0$ represents, as we have just proved, a cylinder generated by lines parallel to **OZ**. The equation $f(x, y, k)=0$ is satisfied at all points where $f(x, y, z)=0$ and $z=k$ are simultaneously satisfied, *i.e.* at all points common to the plane and the locus of the equation $f(x, y, z)=0$, and hence $f(x, y, k)=0$ represents the cylinder generated by lines parallel to **OZ** which pass through the common points, (fig. 6). The two equations $f(x, y, k)=0$, $z=k$ represent the curve of section of the cylinder by the plane $z=k$, which is the curve of section of the locus by the plane $z=k$. If, now, all real values from $-\infty$ to $+\infty$ be given to k, the curve $f(x, y, k)=0$, $z=k$, varies continuously and generates a surface. The coordinates of every point on this surface satisfy the equation $f(x, y, z)=0$, for they satisfy, for some value of k, $f(x, y, k)=0$, $z=k$; and any point (x_1, y_1, z_1) whose coordinates satisfy $f(x, y, z)=0$ lies on the surface, for the coordinates satisfy $f(x, y, z_1)=0$, $z=z_1$, and therefore the point is on one of the curves which generate the surface. Hence the equation $f(x, y, z)=0$ represents a surface, and the surface is the locus of a variable point whose coordinates satisfy the equation.

Ex. 1. Discuss the form of the surface represented by

$$x^2/a^2+y^2/b^2+z^2/c^2=1.$$

The section by the plane $z=k$ has equations

$$z=k, \quad x^2/a^2+y^2/b^2=1-k^2/c^2.$$

The section is therefore a real ellipse if $k^2<c^2$, is imaginary if $k^2>c^2$, and reduces to a point if $k^2=c^2$. The surface is therefore generated by a variable ellipse whose plane is parallel to **XOY** and whose centre is on **OZ**. The ellipse increases from a point in the plane $z=-c$ to the ellipse in the plane **XOY** which is given by $x^2/a^2+y^2/b^2=1$, and then decreases to a point in the plane $z=c$. The surface is the *ellipsoid*, (fig. 29).

Ex. 2. What surfaces are represented by the equations, referred to rectangular axes,

$$\text{(i) } x^2+y^2+z^2=a^2, \quad \text{(ii) } x^2+y^2=2az?$$

Ex. 3. Discuss the forms of the surfaces

$$\text{(i) } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad \text{(ii) } \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

(i) The hyperboloid of one sheet (fig. 30).

(ii) The hyperboloid of two sheets (fig. 31).

Ex. 4. What loci are represented by

(i) $f(u)=0$, (ii) $f(z)=0$, (iii) $f(r, \theta)=0$,
(iv) $f(\theta, \phi)=0$, (v) $f(r, \phi)=0$, (vi) $f(u, \phi)=0$?

Ans. (i) A system of coaxial right cylinders; (ii) a system of planes parallel to XOY; (iii) the surface of revolution generated by rotating the curve in the plane ZOX whose polar equation is $f(r, \theta)=0$ about the z-axis; (iv) a cone whose vertex is at O; (v) a surface generated by circles whose planes pass through OZ and whose dimensions vary as the planes rotate about OZ; (vi) a cylinder whose generators are parallel to OZ, and whose section by the plane $z=0$ is the curve $f(u, \phi)=0$.

10. The equations to a curve. The two equations $f_1(x, y, z)=0$, $f_2(x, y, z)=0$ represent the curve of intersection of the two surfaces given by $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$. If we eliminate one of the variables, z, say, between the two equations, we obtain an equation, $\phi(x, y)=0$, which represents a cylinder whose generators are parallel to OZ. If any values of x, y, z satisfy $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$, they satisfy $\phi(x, y)=0$, and hence the cylinder passes through the curve of intersection of the

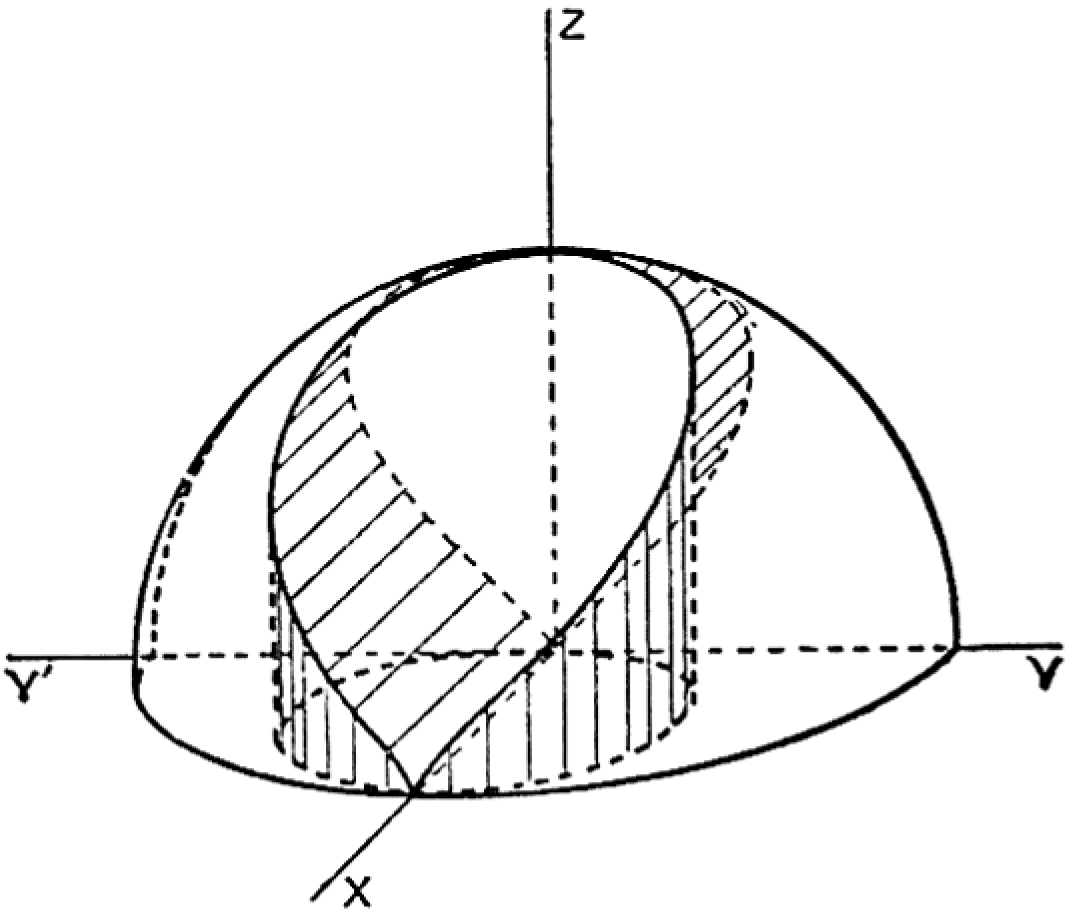

FIG. 7.

Fig. 7 shews part of the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the right circular cylinder $x^2+y^2=ax$. The cylinder which projects the curve on the plane $x=0$ is also shewn. Its equation is $a^2(y^2-z^2)+z^4=0$. The projection of the curve on the plane ZOX is the parabola whose equations are $y=0$, $z^2=a(a-x)$.

surfaces. If the axes are rectangular $\phi(x, y)=0$ represents the cylinder which projects orthogonally the curve of intersection on the plane XOY, and the equations to the projection are $\phi(x, y)=0$, $z=0$.

Ex. 1. If the axes are rectangular, what loci are represented by (i) $x^2+y^2=a^2$, $z^2=b^2$; (ii) $x^2+y^2+z^2=a^2$, $y^2=4az$; (iii) $x^2+y^2=a^2$, $x^2=b^2$, $(a^2>b^2)$?

Ex. 2. Find the equations to the cylinders with generators parallel to OX, OY, OZ, which pass through the curve of intersection of the surfaces represented by $x^2+y^2+2z^2=12$, $x-y+z=1$.

Ans. $2y^2-2yz+3z^2+2y-2z-11=0$, $2x^2+2xz+3z^2-2x-2z-11=0$, $3x^2-4xy+3y^2-4x+4y-10=0$.

11. Surfaces of revolution. Let P, $(0, y_1, z_1)$, (fig. 8), be any point on the curve in the plane YOZ whose Cartesian equation is $f(y, z)=0$. Then

$$f(y_1, z_1)=0 \quad\text{.........................(1)}$$

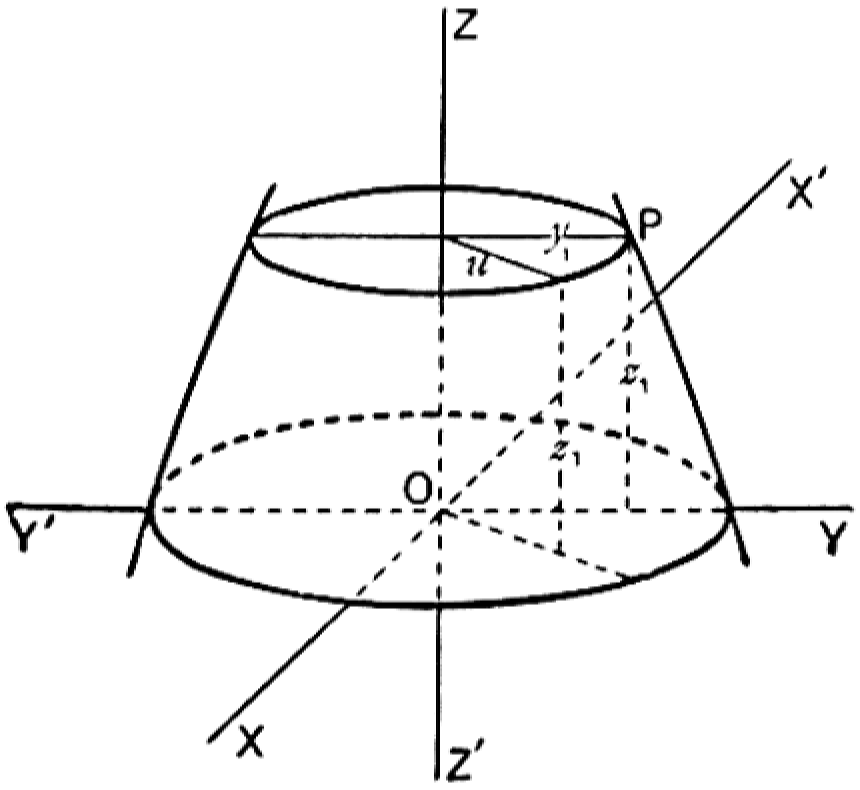

FIG. 8.

The rotation of the curve about OZ produces a surface of revolution. As P moves round the surface, z_1, the z-coordinate of P remains unaltered, and u, the distance of P from the z-axis, is always equal to y_1. Therefore, by (1), the cylindrical coordinates of P satisfy the equation $f(u, z)=0$. But P is any point on the curve, or surface, and therefore the cylindrical equation to the surface is $f(u, z)=0$. Hence the Cartesian equation to the surface is $f(\sqrt{x^2+y^2}, z)=0$.

Since the distance of the point (x, y, z) from the y-axis is $\sqrt{z^2+x^2}$, it follows as before that the equation to the surface formed by rotating the curve $f(y, z)=0$, $x=0$ about OY is $f(y, \sqrt{z^2+x^2})=0$, and similarly $f(\sqrt{y^2+z^2}, x)=0$ represents a surface of revolution whose axis is OX.

Ex. 1. The equation $x^2+y^2+z^2=a^2$ represents the sphere formed by the revolution of the circle $x^2+y^2=a^2$, $z=0$, about OX or OY.

Ex. 2. The surface generated by the revolution of the parabola $y^2=4ax$, $z=0$, about its axis has equation $y^2+z^2=4ax$; about the tangent at the vertex, equation $y^4=16a^2(z^2+x^2)$.

Ex. 3. The surfaces generated by rotating the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, about its axes are given by $\dfrac{x^2}{a^2}+\dfrac{y^2+z^2}{b^2}=1$, $\dfrac{x^2+z^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Ex. 4. Find the equations to the cones formed by rotating the line $z=0$, $y=2x$ about OX and OY.

Ans. $4x^2-y^2-z^2=0$, $4x^2-y^2+4z^2=0$.

Ex. 5. Find the equation to the surface generated by the revolution of the circle $x^2+y^2+2ax+b^2=0$, $z=0$, about the y-axis.

Ans. $(x^2+y^2+z^2+b^2)^2=4a^2(x^2+z^2)$.

Ex. 6. Sketch the forms of the surfaces:

(i) $(y^2+z^2)(2a-x)=x^3$, (ii) $r^2=a^2\cos 2\theta$, (iii) $u^2=2cz$.

The surfaces are generated by rotating (i) the curve $y^2(2a-x)=x^3$ about OX; (ii) the lemniscate in the plane ZOX, $r^2=a^2\cos 2\theta$, about OZ; (iii) the parabola in the plane YOZ, $y^2=2cz$, about OZ.

Ex. 7. Prove that the locus of a point, the sum of whose distances from the points $(a, 0, 0)$, $(-a, 0, 0)$ is constant, $(2k)$, is the ellipsoid of revolution $\dfrac{x^2}{k^2}+\dfrac{y^2+z^2}{k^2-a^2}=1$.

CHAPTER II.

PROJECTIONS.

12. The angle that a given directed line OP makes with a second directed line OX we shall take to be the smallest angle generated by a variable radius turning in the plane XOP from the position OX to the position OP. The sign of the angle is determined by the usual convention. Thus, in figures 9 and 10, θ_1 is the positive angle, and θ_2 the negative angle that OP makes with OX.

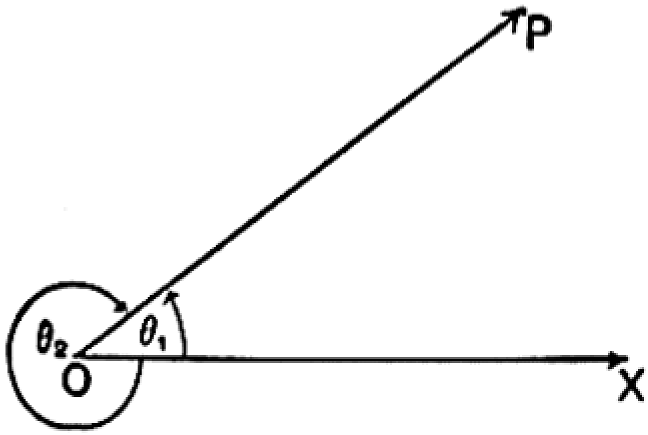

FIG. 9.

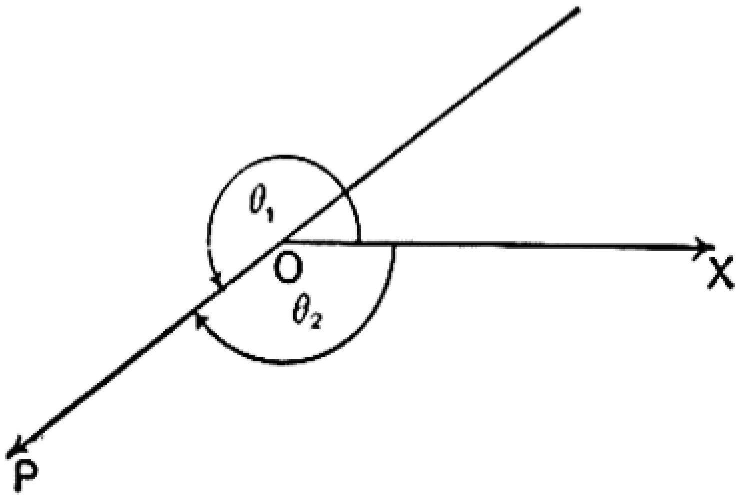

FIG. 10.

13. Projection of a segment. *If* AB *is a given segment and* A′, B′ *are the feet of the perpendiculars from* A, B *to a given line* X′X, *the segment* A′B′ *is the projection of the segment* AB *on* X′X.

From the definition it follows that the projection of BA is B′A′, and therefore that the projections of AB and BA differ only in sign.

It is evident that A′B′ is the intercept made on X′X by the planes through A and B normal to X′X, and hence *the projections of equivalent segments are equivalent segments.*

14. *If* AB *is a given segment of a directed line* MN *whose positive direction,* MN, *makes an angle* θ *with a* *given line* X′X, *the projection of* AB *on* X′X *is equal to* AB . $\cos\theta$.

In figures 11 and 12, AB is positive, in figures 13 and 14, AB is negative.

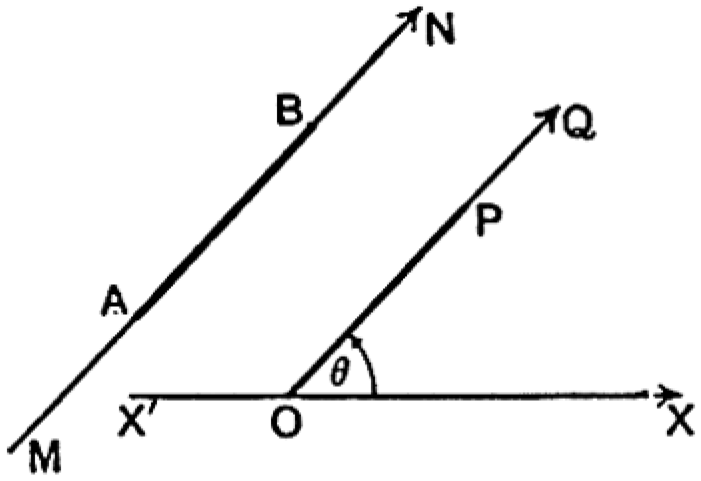

FIG. 11.

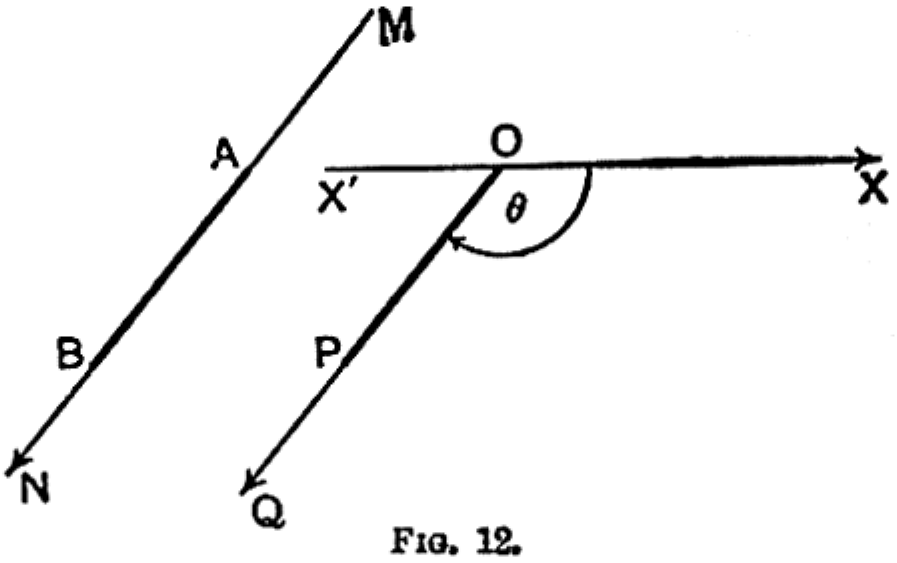

FIG. 12.

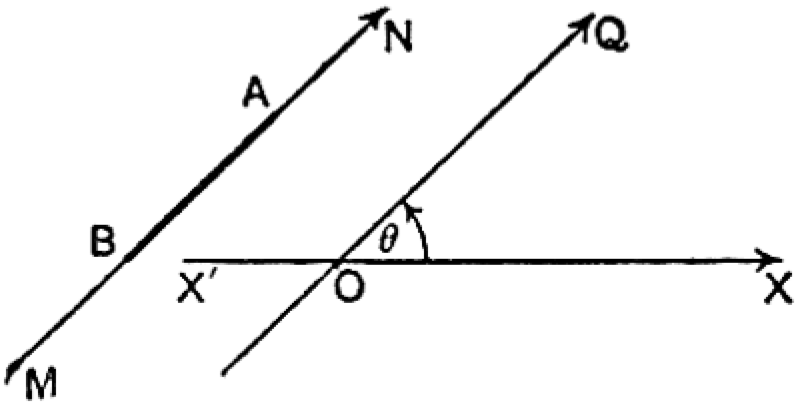

FIG. 13.

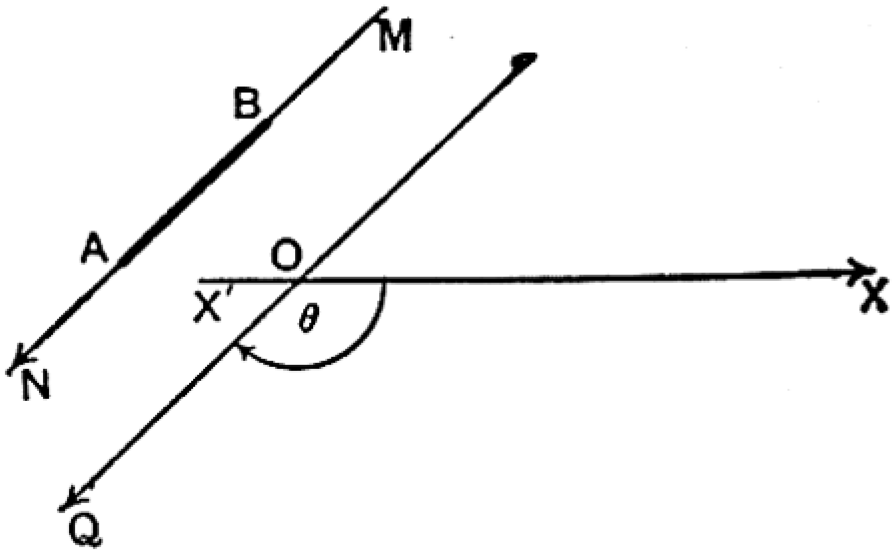

FIG. 14.

Draw OQ from O in the same direction as MN. *If* AB *is positive,* cut off OP, the segment equivalent to AB; then

$$\begin{aligned} \text{the projection of AB} &= \text{the projection of OP}, \\ &= \text{OP} . \cos\theta, \quad \text{(by the definition} \\ &= \text{AB} . \cos\theta. \qquad \text{of cosine),} \end{aligned}$$

If AB *is negative,* BA is positive, and therefore

$$\text{the projection of BA} = \text{BA} . \cos\theta,$$

$$\textit{i.e.} \quad -(\text{the projection of AB}) = -\text{AB} . \cos\theta,$$

$$\textit{i.e.} \quad \text{the projection of AB} = \text{AB} . \cos\theta.$$

15. *If* A, B, C, ... M, N *are any n points in space, the sum of the projections of* AB, BC, ... MN, *on any given line* X′X *is equal to the projection of the straight line* AN *on* X′X.

Let the feet of the perpendiculars from A, B, ... M, N, to X′X be A′, B′, ... M′, N′. Then, (§ 2),

$$A'B' + B'C' + \ldots M'N' = A'N',$$

which proves the proposition

16. The angle between two planes we shall take to be the angle that the positive direction of a normal to one makes with the positive direction of a normal to the other.

17. Projection of a closed plane figure. *If the projections of three points* A, B, C *on a given plane are* A′, B′, C′, *then* $\triangle A'B'C' = \cos\theta \triangle ABC$, *where* θ *is the angle between the planes* ABC, A′B′C′.

Consider first the areas ABC, A′B′C′ without regard to sign.

(i) If the planes ABC, A′B′C′ are parallel, the equation $\triangle A'B'C' = \cos\theta \triangle ABC$ is obviously true.

(ii) If one side of the triangle ABC, say BC, is parallel to the plane A′B′C′, let AA′ meet the plane through BC parallel to the plane A′B′C′ in A_2, (fig. 15). Draw A_2D at right angles to BC, and join AD. Then BC is at right angles to A_2D and AA_2, and therefore BC is normal to the plane AA_2D and therefore at right angles to AD. Hence the angle A_2DA is equal to θ, or its supplement.

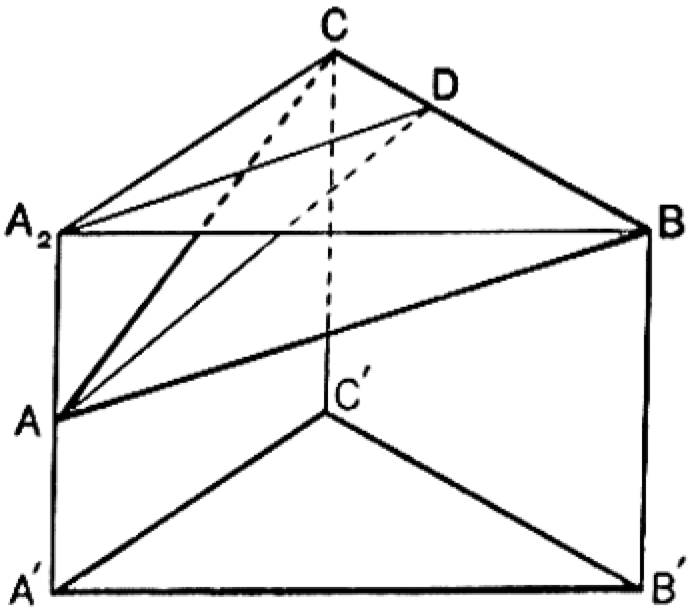

FIG. 15.

But $$\triangle A'B'C' = \triangle A_2BC,$$

and $$\triangle A_2BC : \triangle ABC = A_2D : AD = \cos\angle A_2DA;$$

therefore $$\triangle A'B'C' = \cos\theta \triangle ABC.$$

(iii) If none of the sides of the triangle ABC is parallel to the plane A′B′C′, draw lines through A, B, C parallel to the line of intersection of the planes ABC, A′B′C′. These lines lie in the plane ABC and are parallel to the plane A′B′C′, and one of them, that through A, say, will cut the opposite side, BC, of the triangle ABC, internally. And therefore the triangle ABC can always be divided by a line through a vertex into two triangles, with a common side parallel to the given plane A′B′C′, and hence, by (ii), $\triangle A'B'C' = \cos\theta \, \triangle ABC$.

Suppose now that the areas ABC, A′B′C′ are considered positive or negative according as the directions of rotation given by ABC, A′B′C′ are positive or negative. Then, applying the convention of § 4 to figures 16 and 17, we

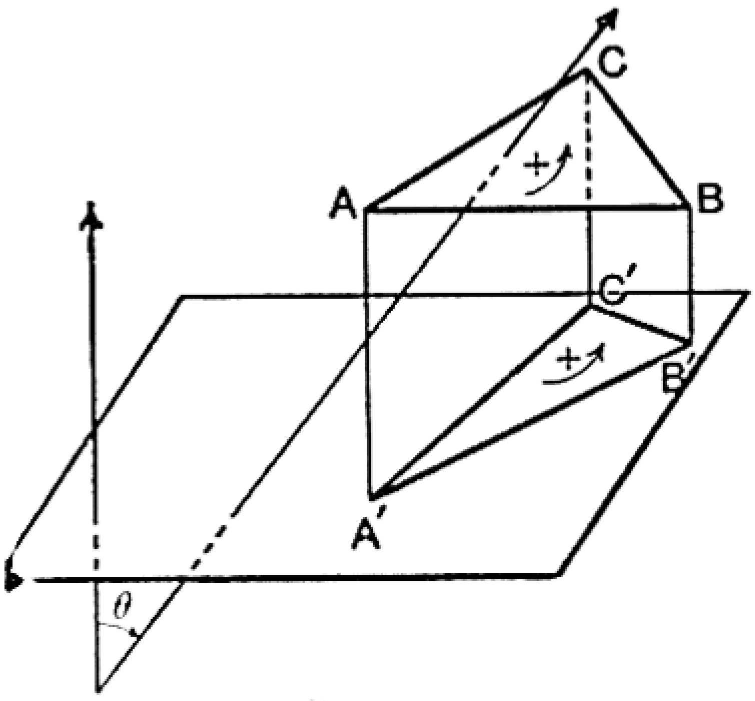

FIG. 16.

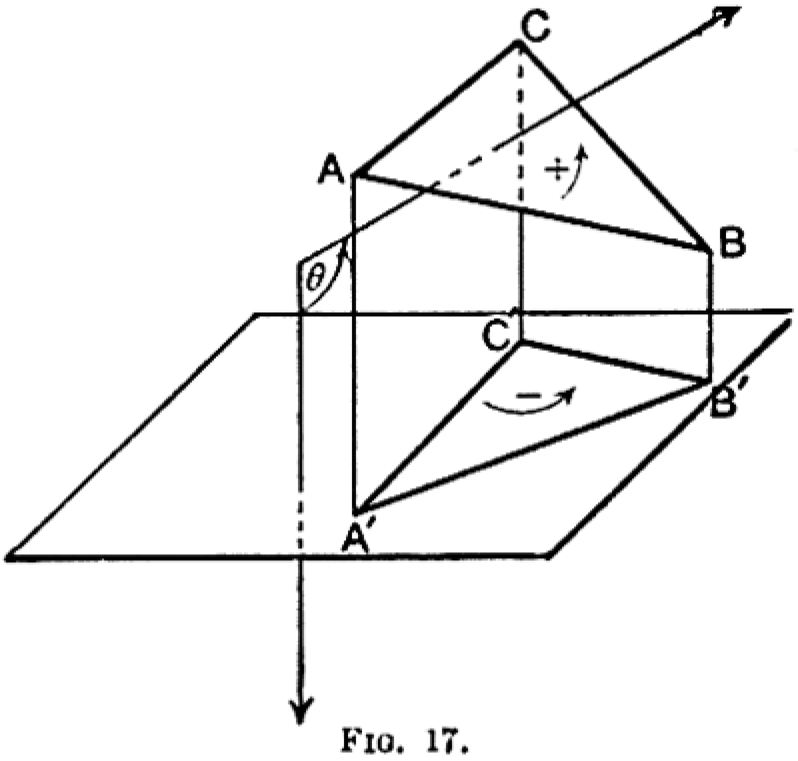

FIG. 17.

see that if $\cos\theta$ is positive, the directions of rotation ABC, A′B′C′ have the same sign, and that if $\cos\theta$ is negative, they have opposite signs. That is, the areas have the same sign if $\cos\theta$ is positive, and opposite signs if $\cos\theta$ is negative. Hence the equation $\triangle A'B'C' = \cos\theta \, \triangle ABC$ is true for the signs as well as the magnitudes of the areas.

18. *If* A, B, C, … N *are any coplanar points and* A′, B′, C′, … N′ *are their projections on any given plane, then*

$$\textit{area } A'B'C' \ldots N' : \textit{area } ABC \ldots N = \cos\theta,$$

where θ *is the angle between the planes.*

Let O be any point of the plane ABC … N, and O′ be its projection on the plane A′B′C′ … N′.

Then $\quad$ area $ABC \ldots N = \triangle OAB + \triangle OBC + \ldots \triangle ONA$,

and $\quad$ area $A'B'C' \ldots N' = \triangle O'A'B' + \triangle O'B'C' + \ldots \triangle O'N'A'$.

But $\triangle O'A'B' = \cos\theta \, \triangle OAB$, etc., and therefore the result follows.

19. *If* A_0 *is the area of any plane curve and* A *is the area of its projection on any given plane,* $A = \cos\theta \, . \, A_0$, *where* θ *is the angle between the planes.*

For A_0 is the limit, as n tends to infinity, of the area of an inscribed n-gon, and A is the limit of the area of the projection of the n-gon, and, by § 18, the ratio of these areas is $\cos\theta$.

Ex. 1. AA' is a diameter of a given circle, and P is a plane through AA' making an angle θ with the plane of the circle. If B is any point on the circle and B' is its projection on the plane P, the perpendiculars from B and B' to AA' are in the constant ratio $1 : \cos\theta$, and the projection is therefore a curve such that its ordinate to AA' is in a constant ratio to the corresponding ordinate of the circle; that is, the projection is an ellipse whose major axis is AA' and whose auxiliary circle is equal to the given circle. The minor axis is $\cos\theta \, . \, AA'$; therefore if $AA' = 2a$ and $\cos\theta = b/a$, the minor axis is $2b$. By § 19, the area of the ellipse $= \cos\theta \, . \, \pi a^2 = \pi ab$.

Ex. 2. Find the area of the section of the cylinder $16x^2 + 9y^2 = 144$ by a plane whose normal makes an angle of 60° with OZ. *Ans.* 24π

DIRECTION-COSINES.

20. If α, β, γ are the angles that a given directed line makes with the positive directions $X'OX$, $Y'OY$, $Z'OZ$ of the coordinate axes, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the **direction-cosines** of the line.

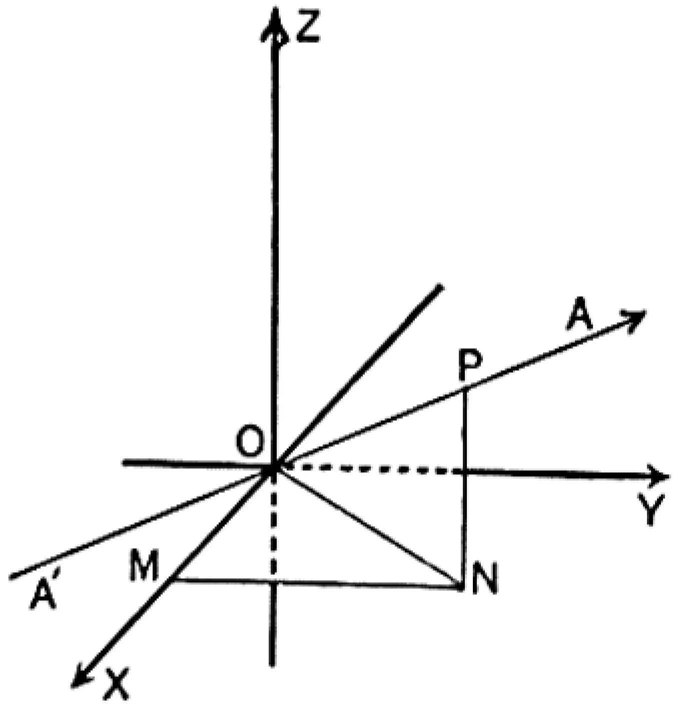

Fig. 18.

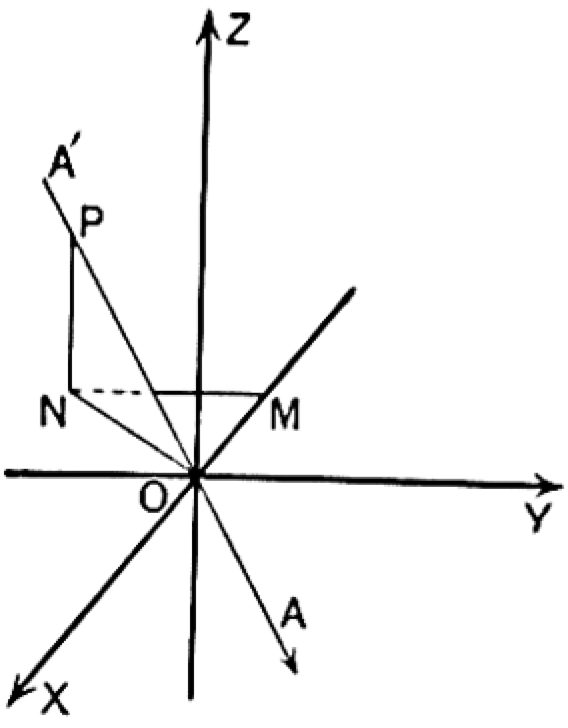

Fig. 19.

21. Direction-cosines referred to rectangular axes. Let $A'OA$ be the line through O which has direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Let P, (x, y, z) be any point on $A'OA$, and **OP** have measure r. In fig. 18, r is positive; in fig. 19, r is negative. Draw **PN** perpendicular to the plane **XOY**, and **NM** in the plane **XOY**, perpendicular to **OX**. Then the measures of **OM**, **MN**, **NP** are x, y, z respectively. Since **OM** is the projection of **OP** on **OX**,

$$x=r\cos\alpha, \text{ and similarly, } y=r\cos\beta,\ z=r\cos\gamma. \quad \text{....(1)}$$

Again the projection of **OP** on any line is equal to the sum of the projections of **OM**, **MN**, **NP**, and therefore, projecting on **OP**, we obtain

$$r=x\cos\alpha+y\cos\beta+z\cos\gamma. \quad \text{................(2)}$$

But $x/r=\cos\alpha$, $y/r=\cos\beta$, $z/r=\cos\gamma$; therefore

$$1=\cos^2\alpha+\cos^2\beta+\cos^2\gamma. \quad \text{................(3)}$$

This is the formula in three dimensions which corresponds to $\cos^2\theta+\sin^2\theta=1$ in plane trigonometry.

Cor. 1. By substituting for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in (2) or (3), we obtain $r^2=x^2+y^2+z^2$, (cf. § 6, Cor.).

Cor. 2. If (x, y, z) is any point on the line through **O** whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, we have, by (1),

$$\frac{x}{\cos\alpha}=\frac{y}{\cos\beta}=\frac{z}{\cos\gamma},\ (=r).$$

Cor. 3. If (x, y, z) is any point on the line through (x_1, y_1, z_1) whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, by changing the origin we obtain

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}.$$

Ex. 1. Prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

Ex. 2. If **P** is the point (x_1, y_1, z_1), prove that the projection of **OP** on a line whose direction-cosines are l_1, m_1, n_1 is $l_1x_1+m_1y_1+n_1z_1$.

The projection of **OP** = projn. of **OM** + projn. of **MN** + projn. of **NP**, (figs. 18, 19),

$$=l_1x_1+m_1y_1+n_1z_1.$$

Ex. 3. If **P**, **Q** are the points (x_1, y_1, z_1), (x_2, y_2, z_2), prove that the projection of **PQ** on a line whose direction-cosines are l_1, m_1, n_1 is

$$l_1(x_2-x_1)+m_1(y_2-y_1)+n_1(z_2-z_1).$$

(Change the origin to **P** and apply Ex. 2.)

Ex. 4. The projections of a line on the axes are 2, 3, 6. What is the length of the line? *Ans.* 7.

Ex. 5. A plane makes intercepts **OA**, **OB**, **OC**, whose measures are a, b, c, on the axes **OX**, **OY**, **OZ**. Find the area of the triangle **ABC**.

Let the positive direction of the normal from **O** to the plane **ABC** have direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and let Δ denote the area **ABC**. Then since $\triangle$**OBC** is the projection of $\triangle$**ABC** on the plane **YOZ**, $\cos\alpha \,.\, \Delta=\frac{1}{2}bc$, and similarly, $\cos\beta \,.\, \Delta=\frac{1}{2}ca$, $\cos\gamma \,.\, \Delta=\frac{1}{2}ab$. Therefore, since

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1, \quad \Delta=\tfrac{1}{2}\{b^2c^2+c^2a^2+a^2b^2\}^{\frac{1}{2}}.$$

Ex. 6. Find the areas of the projections of the curve $x^2+y^2+z^2=25$, $x+2y+2z=9$ on the coordinate planes, and having given that the curve is plane, find its area.

(Cf. Ex. 2, § 10.) *Ans.* $16\pi/3$, $32\pi/3$, $32\pi/3$; 16π.

22. If a, b, c are given proportionals to the direction-cosines of a line, the actual direction-cosines are found from the relations

$$\frac{\cos\alpha}{a}=\frac{\cos\beta}{b}=\frac{\cos\gamma}{c}=\frac{\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}}{\sqrt{a^2+b^2+c^2}}=\frac{\pm 1}{\sqrt{a^2+b^2+c^2}}.$$

If **P** is the point (a, b, c) and the direction-cosines of the directed line **OP** are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, then, since **OP** is positive and equal to $\sqrt{a^2+b^2+c^2}$,

$$\cos\alpha=\frac{a}{\mathsf{OP}}=\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad \cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$\cos\gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

The direction-cosines of **PO** are

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}, \quad \frac{-b}{\sqrt{a^2+b^2+c^2}}, \quad \frac{-c}{\sqrt{a^2+b^2+c^2}}.$$

Ex. 1. Find the direction-cosines of a line that makes equal angles with the axes.

Ans. $\cos\alpha=\cos\beta=\cos\gamma=\pm 1/\sqrt{3}$; (whence the acute angles which the line makes with the axes are equal to 54° 44′).

Ex. 2. **P** and **Q** are $(2, 3, -6)$, $(3, -4, 5)$. Find the direction-cosines of **OP**, **OQ**, **PO**.

Ans. $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$; $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$; $\frac{-2}{7}, \frac{-3}{7}, \frac{6}{7}$.

Ex. 3. If **P**, **Q** are (x_1, y_1, z_1), (x_2, y_2, z_2) the direction-cosines of **PQ** are proportional to x_2-x_1, y_2-y_1, z_2-z_1.

Ex. 4. If P, Q are (2, 3, 5), (−1, 3, 2), find the direction-cosines of PQ.
Ans. $\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$.

Ex. 5. If P, Q, R, S are the points (3, 4, 5), (4, 6, 3), (−1, 2, 4), (1, 0, 5), find the projection of RS on PQ. *Ans.* $-\frac{4}{3}$.

Ex. 6. If P, Q, R, S are the points (2, 3, −1), (3, 5, −3), (1, 2, 3), (3, 5, 7), prove by projections that PQ is at right angles to RS.

23. The angle between two lines. *If* OP *and* OQ *have direction-cosines* $\cos\alpha$, $\cos\beta$, $\cos\gamma$; $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, *and* θ *is the angle that* OP *makes with* OQ,

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

If, as in § 21, P is (x, y, z) and the measure of OP is r, projecting OP and OM, MN, NP on OQ, we obtain

$$r\cos\theta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'.$$

But $\quad x = r\cos\alpha,\ y = r\cos\beta,\ z = r\cos\gamma;$

therefore $\quad \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$

Cor. 1. We have the identity

$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$\equiv (mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2.$$

(This identity is known as *Lagrange's identity.* We shall frequently find it advantageous to apply it.)

Hence

$$\sin^2\theta = (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)(\cos^2\alpha'+\cos^2\beta'+\cos^2\gamma')$$
$$-(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')^2,$$
$$=(\cos\beta\cos\gamma'-\cos\gamma\cos\beta')^2+(\cos\gamma\cos\alpha'-\cos\alpha\cos\gamma')^2$$
$$+(\cos\alpha\cos\beta'-\cos\beta\cos\alpha')^2.$$

Cor. 2. If θ is an angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c',

$$\cos\theta = \frac{\pm(aa'+bb'+cc')}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}},$$

and $\quad \sin\theta = \frac{\pm\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$

Cor. 3. If the lines are at right angles,
$\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'=0$, or $aa'+bb'+cc'=0$.

Cor. 4. If the lines are parallel,

$$\cos\beta\cos\gamma' - \cos\gamma\cos\beta' = 0, \quad \cos\gamma\cos\alpha' - \cos\alpha\cos\gamma' = 0,$$

and
$$\cos\alpha\cos\beta' - \cos\beta\cos\alpha' = 0,$$

whence $\cos\alpha = \cos\alpha'$, $\cos\beta = \cos\beta'$, and $\cos\gamma = \cos\gamma'$ (as is evident from the definition of direction-cosines); or

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Ex. 1. If **P**, **Q** are $(2, 3, -6)$, $(3, -4, 5)$, find the angle that **OP** makes with **OQ**.

Ans. $\cos\theta = \dfrac{-18\sqrt{2}}{35}$.

Ex. 2. **P**, **Q**, **R** are $(2, 3, 5)$, $(-1, 3, 2)$, $(3, 5, -2)$. Find the angles of the triangle **PQR**.

Ans. 90°, $\cos^{-1}\dfrac{\sqrt{6}}{3}$, $\cos^{-1}\dfrac{\sqrt{3}}{3}$.

Ex. 3. Find the angles between the lines whose direction-cosines are proportional to (i) $2, 3, 4$; $3, 4, 5$; (ii) $2, 3, 4$; $1, -2, 1$.

Ans. (i) $\cos^{-1}\dfrac{38}{5\sqrt{58}}$, (ii) 90°.

Ex. 4. The lines whose direction-cosines are proportional to $2, 1, 1$; $4, \sqrt{3}-1, -\sqrt{3}-1$; $4, -\sqrt{3}-1, \sqrt{3}-1$ are inclined to one another at an angle $\pi/3$.

Ex. 5. If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, the line whose direction-cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ makes equal angles with them.

Ex. 6. Find the angle between two diagonals of a cube.

Ans. $\cos^{-1} 1/3$.

Ex. 7. Prove by direction-cosines that the points $(3, 2, 4)$, $(4, 5, 2)$, $(5, 8, 0)$, $(2, -1, 6)$ are collinear.

Ex. 8. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube; prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3.$$

Ex. 9. If the edges of a rectangular parallelepiped are a, b, c, shew that the angles between the four diagonals are given by

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right).$$

Ex. 10. If a variable line in two adjacent positions has direction-cosines l, m, n; $l+\delta l, m+\delta m, n+\delta n$, shew that the small angle, $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

We have $\Sigma l^2 = 1$ and $\Sigma(l+\delta l)^2 = 1$, therefore $\Sigma(\delta l)^2 = -2\Sigma l\,\delta l$.

But
$$\cos\delta\theta = \Sigma l(l+\delta l) = 1 + \Sigma l\,\delta l.$$

Therefore
$$2\sin^2\frac{\delta\theta}{2} = -\Sigma l\,\delta l = \frac{1}{2}\Sigma(\delta l)^2.$$

That is, since
$$\sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2},\ \delta\theta^2 = \Sigma(\delta l)^2.$$

Ex. 11. Lines OA, OB are drawn from O with direction-cosines proportional to $(1, -2, -1)$, $(3, -2, 3)$. Find the direction-cosines of the normal to the plane AOB.

Ans. $\dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}, \dfrac{-2}{\sqrt{29}}$.

Ex. 12. Prove that the three lines drawn from O with direction-cosines proportional to $(1, -1, 1)$, $(2, -3, 0)$, $(1, 0, 3)$ lie in one plane.

Ex. 13. Prove that the three lines drawn from O with direction-cosines l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, are coplanar if

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0.$$

Ex. 14. Find the direction-cosines of the axis of the right circular cone which passes through the lines drawn from O with direction-cosines proportional to $(3, 6, -2)$, $(2, 2, -1)$, $(-1, 2, 2)$, and prove that the cone also passes through the coordinate axes.

Ans. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Ex. 15. Lines are drawn from O with direction-cosines proportional to $(1, 2, 2)$, $(2, 3, 6)$, $(3, 4, 12)$. Prove that the axis of the right circular cone through them has direction-cosines $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$, and that the semivertical angle of the cone is $\cos^{-1} 1/\sqrt{3}$.

24. Distance of a point from a line. *To find the distance of* P, (x', y', z') *from the line through* A, (a, b, c), *whose direction-cosines are* $\cos\alpha, \cos\beta, \cos\gamma$.

Let PN, the perpendicular from P to the line, have measure δ. Then AN is the projection of AP on the line, and its measure is, (Ex. 3, § 21),

$$(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma.$$

But $$PN^2=AP^2-AN^2,$$

therefore

$$\begin{aligned}\delta^2=\{(x'-a)^2+(y'-b)^2+(z'-c)^2\}(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\\ -\{(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma\}^2,\end{aligned}$$

which, by Lagrange's identity, gives

$$\begin{aligned}\delta^2=&\{(y'-b)\cos\gamma-(z'-c)\cos\beta\}^2\\ &+\{(z'-c)\cos\alpha-(x'-a)\cos\gamma\}^2\\ &+\{(x'-a)\cos\beta-(y'-b)\cos\alpha\}^2.\end{aligned}$$

Cor. If (x', y', z') is any point on the line, $\delta=0$, and

$$\frac{x'-a}{\cos\alpha}=\frac{y'-b}{\cos\beta}=\frac{z'-c}{\cos\gamma}. \quad \text{(Cf. § 21, Cor. 3.)}$$

Ex. 1. Find the distance of $(-1, 2, 5)$ from the line through $(3, 4, 5)$ whose direction-cosines are proportional to $2, -3, 6$.

Ans. $\dfrac{4\sqrt{61}}{7}$.

Ex. 2. Find the distance of A, $(1, -2, 3)$ from the line, PQ, through P, $(2, -3, 5)$, which makes equal angles with the axes.

Ans. $\sqrt{\dfrac{14}{3}}$.

Ex. 3. Shew that the equation to the right circular cone whose vertex is at the origin, whose axis has direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and whose semivertical angle is θ, is

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2$$
$$= \sin^2\theta(x^2+y^2+z^2).$$

Ex. 4. Find the equation to the right circular cone whose vertex is P, axis PQ (Ex. 2), and semivertical angle is 30°.

Ans. $4\{(y-z+8)^2+(z-x-3)^2+(x-y-5)^2\}$
$$=3\{(x-2)^2+(y+3)^2+(z-5)^2\}.$$

Ex. 5. Find the equation to the right circular cone whose vertex is P, axis PQ, and which passes through A (Ex. 2).

Ans. $3\{(y-z+8)^2+(z-x-3)^2+(x-y-5)^2\}$
$$=7\{(x-2)^2+(y+3)^2+(z-5)^2\}.$$

Ex. 6. The axis of a right cone, vertex O, makes equal angles with the coordinate axes, and the cone passes through the line drawn from O with direction-cosines proportional to $(1, -2, 2)$. Find the equation to the cone. *Ans.* $4x^2+4y^2+4z^2+9yz+9zx+9xy=0$.

Ex. 7. Find the equation to the right circular cylinder of radius 2 whose axis passes through $(1, 2, 3)$ and has direction-cosines proportional to $(2, -3, 6)$.

Ans. $9(2y+z-7)^2+4(z-3x)^2+(3x+2y-7)^2=196$.

***25. Direction-cosines referred to oblique axes.** Let X′OX, Y′OY, Z′OZ, (fig. 20), be oblique axes, the angles YOZ, ZOX, XOY being λ, μ, ν respectively. Let A′OA be the line through O whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Take P, (x, y, z) any point on A′OA, and let the measure of OP be r. Draw PN parallel to OZ to meet the plane XOY in N, and NM parallel to OY to meet OX in M. Then, since the projection of OP is equal to the sum of the projections of **OM, MN, NP**, projecting on **OX, OY, OZ, OP** in turn, we obtain

$$r\cos\alpha = x + y\cos\nu + z\cos\mu, \quad \ldots\ldots\ldots\ldots(1)$$

$$r\cos\beta = x\cos\nu + y + z\cos\lambda, \quad \ldots\ldots\ldots\ldots(2)$$

$$r\cos\gamma = x\cos\mu + y\cos\lambda + z, \quad \ldots\ldots\ldots\ldots(3)$$

$$r = x\cos\alpha + y\cos\beta + z\cos\gamma. \ldots\ldots\ldots(4)$$

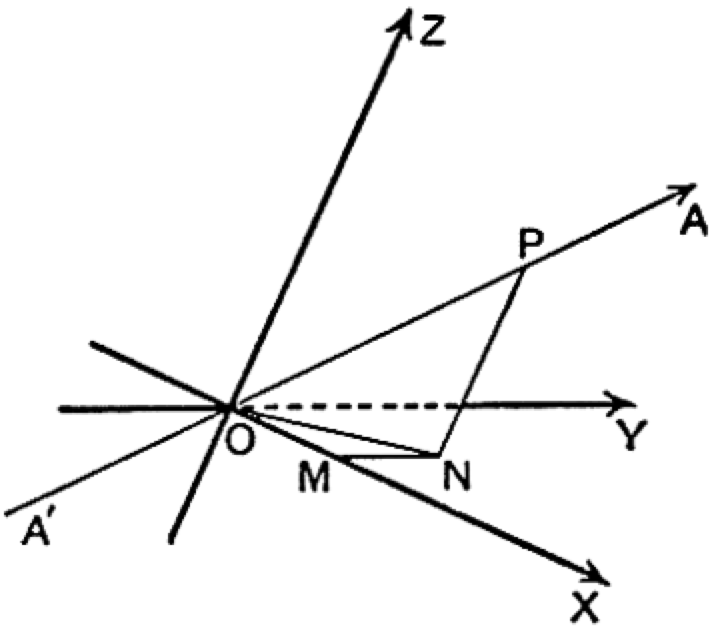

FIG. 20.

Therefore, eliminating r, x, y, z, we have the relation satisfied by the direction-cosines of any line

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma, & 1 \end{vmatrix} = 0,$$

which may be written,

$$\Sigma\sin^2\lambda\cos^2\alpha - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)\cos\beta\cos\gamma$$
$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu.$$

Cor. 1. Multiply (1), (2), (3) by x, y, z respectively, and add, then

$$x^2 + y^2 + z^2 + 2yz\cos\lambda + 2zx\cos\mu + 2xy\cos\nu$$
$$= r(x\cos\alpha + y\cos\beta + z\cos\gamma),$$
$$= r^2, \text{ [by (4)]}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{A})$$

Cor. 2. If **P, Q** are (x_1, y_1, z_1), (x_2, y_2, z_2), **PQ**^{2} is given by

$$\Sigma(x_2 - x_1)^2 + 2\Sigma(y_2 - y_1)(z_2 - z_1)\cos\lambda.$$

Ex. 1. If **P**, (x, y, z) is any point on the plane through **O** at right angles to **OX**, the projection of **OP** on **OX** is zero, and therefore

$$x + y\cos\nu + z\cos\mu = 0.$$

Ex. 2. If **P**, (x, y, z) is any point on the normal through **O** to the plane **XOY**, $x + y\cos\nu + z\cos\mu = 0 = x\cos\nu + y + z\cos\lambda$.

***26.** If a, b, c are given proportionals to the direction-cosines of a line, the actual direction-cosines are given by

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}$$

$$= \frac{\pm\{\Sigma\sin^2\lambda\cos^2\alpha - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)\cos\beta\cos\gamma)^{\frac{1}{2}}}{\{\Sigma\sin^2\lambda\,.\,a^2 - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)bc\}^{\frac{1}{2}}}$$

$$= \frac{\pm\{1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu\}^{\frac{1}{2}}}{\{\Sigma\sin^2\lambda\,.\,a^2 - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)bc\}^{\frac{1}{2}}}.$$

***27. The angle between two lines.** If **OQ** has direction-cosines $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, and makes an angle θ with **OP**, projecting on **OQ**, we obtain

$$r\cos\theta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'. \quad\ldots\ldots\ldots\ldots(5)$$

Therefore eliminating x, y, z, r between equations (1), (2), (3) of § 25, and (5), we have

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha', & \cos\beta', & \cos\gamma', & \cos\theta \end{vmatrix} = 0, \quad \text{or}$$

$$\Sigma(\sin^2\lambda\cos\alpha\cos\alpha') - \Sigma\{(\cos\lambda - \cos\mu\cos\nu) \times (\cos\beta\cos\gamma' + \cos\beta'\cos\gamma)\}$$
$$= \cos\theta(1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu).$$

Cor. The angles between the lines whose direction-cosines are proportional to a, b, c; a', b', c' are given by

$$\cos\theta = \frac{\pm\{\Sigma(aa'\sin^2\lambda) - \Sigma(bc' + b'c)(\cos\lambda - \cos\mu\cos\nu)\}}{\{\Sigma a^2\sin^2\lambda - 2\Sigma bc(\cos\lambda - \cos\mu\cos\nu)\}^{\frac{1}{2}} \times \{\Sigma a'^2\sin^2\lambda - 2\Sigma b'c'(\cos\lambda - \cos\mu\cos\nu)\}^{\frac{1}{2}}}.$$

Ex. 1. If $\lambda = \mu = \nu = \pi/3$, find the angles between the lines whose direction-cosines are proportional to

(i) 2, 3, 4 ; 3, 4, 5 ; (ii) 2, 3, 4 ; 1, −2, 1.

Ans. (i) $\cos^{-1}\dfrac{22}{7\sqrt{10}}$; (ii) $\pi/2$.

Ex. 2. Prove that the lines whose direction-cosines are proportional to l, m, n; $m-n, n-l, l-m$ are at right angles if $\lambda=\mu=\nu$.

Ex. 3. The edges OA, OB, OC of a tetrahedron are of lengths a, b, c, and the angles BOC, COA, AOB are λ, μ, ν; find the volume.

Take OA, OB, OC as axes, and draw CN at right angles to the plane AOB. Then if CN is of length p, and V denotes the volume, $\mathrm{V}=\frac{1}{3}\cdot\frac{ab\sin\nu}{2}\cdot p$, and $p=c\cos\angle\mathrm{OCN}$. But the direction-cosines of CN are 0, 0, $\cos\angle\mathrm{OCN}$, therefore, by § 25,

$$\sin^2\nu\cos^2\angle\mathrm{OCN}=1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu,$$

$$\therefore\ \mathrm{V}=\frac{abc}{6}\{1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu\}^{\frac{1}{2}}.$$

DIRECTION-RATIOS.

28. Let OL be drawn from O in the same direction as a given directed line PQ and of unit length. Then the coordinates of L evidently depend only on the direction of PQ, and when given, determine that direction. They are therefore called the **direction-ratios** of PQ.

If the axes are rectangular the direction-ratios are the same as the direction-cosines.

29. *If* P, (x, y, z) *is any point on a given line* A′OA *whose direction-ratios are* l, m, n, *and the measure of* OP *is* r, *then*

$$l=\frac{x}{r},\quad m=\frac{y}{r},\quad n=\frac{z}{r}.$$

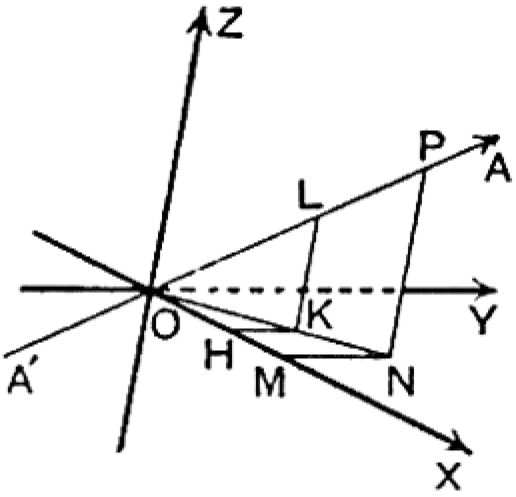

FIG. 21.

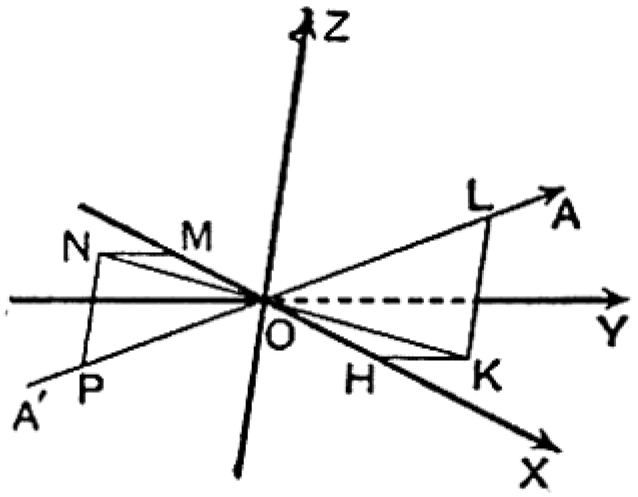

FIG. 22.

In fig. 21, r is positive, in fig. 22 r is negative. LK, PN are parallel to OZ; KH, NM are parallel to OY. Then since the parallel planes PNM, LKH cut X′OX, A′OA proportionally,

$$\mathrm{OP}:\mathrm{OL}=\mathrm{OM}:\mathrm{OH},$$

where OP, OL, OM, OH are directed segments.

But the measures of OM and OH are x and l respectively, and therefore $l = x/r$. Similarly, $m = y/r$, $n = z/r$.

Cor. 1. If P, (x, y, z) is any point on the line through O whose direction-ratios are l, m, n,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad \text{(Cf. § 21, Cor. 2.)}$$

Cor. 2. If (x, y, z) is any point on the line through (x', y', z') whose direction-ratios are l, m, n,

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n}. \quad \text{(Cf. § 21, Cor. 3.)}$$

Cor. 3. If P, Q are (x_1, y_1, z_1), (x_2, y_2, z_2), and the measure of PQ is r, the direction-ratios of PQ are

$$\frac{x_2 - x_1}{r}, \quad \frac{y_2 - y_1}{r}, \quad \frac{z_2 - z_1}{r}.$$

Ex. 1. Find the direction-ratios of the lines bisecting the angles between the lines whose direction-ratios are l_1, m_1, n_1; l_2, m_2, n_2.

If L, L′ are (l_1, m_1, n_1), (l_2, m_2, n_2), then OL and OL′ are the lines from O with the given direction-ratios, and OL and OL′ are of unit length.

The mid-point, M, of LL′ has coordinates, $\frac{l_1 + l_2}{2}, \frac{m_1 + m_2}{2}, \frac{n_1 + n_2}{2}$, and $\text{OM} = \cos\frac{\theta}{2}$, where $\angle \text{LOL}' = \theta$, therefore the direction-ratios of OM are

$$\frac{l_1 + l_2}{2 \cos \theta/2}, \quad \frac{m_1 + m_2}{2 \cos \theta/2}, \quad \frac{n_1 + n_2}{2 \cos \theta/2}.$$

Similarly, the direction-ratios of the other bisector are $\frac{l_1 - l_2}{2 \sin \theta/2}$, etc.

Ex. 2. OX, OY, OZ are given rectangular axes;
$\text{OX}_1, \text{OY}_1, \text{OZ}_1$ bisect the angles YOZ, ZOX, XOY;
$\text{OX}_2, \text{OY}_2, \text{OZ}_2$ bisect the angles $\text{Y}_1\text{OZ}_1, \text{Z}_1\text{OX}_1, \text{X}_1\text{OY}_1$.

Prove that $\angle \text{Y}_1\text{OZ}_1 = \angle \text{Z}_1\text{OX}_1 = \angle \text{X}_1\text{OY}_1 = \pi/3$, and that

$$\angle \text{Y}_2\text{OZ}_2 = \angle \text{Z}_2\text{OX}_2 = \angle \text{X}_2\text{OY}_2 = \cos^{-1} 5/6.$$

Ex. 3. A, B, C, are the points $(1, 2, 3)$, $(3, 5, -3)$, $(-2, 6, 15)$, and the axes are rectangular. Find the direction-cosines of the interior bisector of the angle BAC. *Ans.* $1/\sqrt{182}$, $67/5\sqrt{182}$, $6/5\sqrt{182}$.

***30.** The direction-ratios of any line satisfy the equation (§ 25, Cor. 1, (A)),

$$l^2 + m^2 + n^2 + 2mn \cos \lambda + 2nl \cos \mu + 2lm \cos \nu = 1,$$

which it is convenient to write, $\phi(l, m, n) = 1$.

***31.** *To find the direction-cosines of the line whose direction-ratios are l, m, n.*

Project **OL**, (figs. 21 and 22), on the axes and on itself, and we obtain, as in § 25 (1), (2), (3), (4),

$$\cos\alpha = l + m\cos\nu + n\cos\mu \equiv \frac{1}{2}\frac{\partial\phi}{\partial l},$$

$$\cos\beta = l\cos\nu + m + n\cos\lambda \equiv \frac{1}{2}\frac{\partial\phi}{\partial m},$$

$$\cos\gamma = l\cos\mu + m\cos\lambda + n \equiv \frac{1}{2}\frac{\partial\phi}{\partial n},$$

$$1 = l\cos\alpha + m\cos\beta + n\cos\gamma. \quad \text{(Cf. § 21 (3).)}$$

***32.** *To find the angles between the lines whose direction-ratios are l, m, n; l', m', n'.*

Let **OL′**, the unit ray from **O** which has direction-ratios l', m', n', make an angle θ with **OL**. Then projecting **OL′** on **OL**, we obtain,

$$\begin{aligned}
\cos\theta &= l'\cos\alpha + m'\cos\beta + n'\cos\gamma,\\
&= \frac{1}{2}\left(l'\frac{\partial\phi}{\partial l} + m'\frac{\partial\phi}{\partial m} + n'\frac{\partial\phi}{\partial n}\right), \quad \text{(by § 31)},\\
&= ll' + mm' + nn' + (mn' + m'n)\cos\lambda\\
&\qquad + (nl' + n'l)\cos\mu + (lm' + l'm)\cos\nu,\\
&= \frac{1}{2}\left(l\frac{\partial\phi}{\partial l'} + m\frac{\partial\phi}{\partial m'} + n\frac{\partial\phi}{\partial n'}\right).
\end{aligned}$$

Cor. If the lines are at right angles,

$$l\frac{\partial\phi}{\partial l'} + m\frac{\partial\phi}{\partial m'} + n\frac{\partial\phi}{\partial n'} = 0,$$

which may be written in the forms,

$$l'\cos\alpha + m'\cos\beta + n'\cos\gamma = 0$$

$$\text{or} \quad l\cos\alpha' + m\cos\beta' + n\cos\gamma = 0,$$

where $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ are the direction-cosines of **OL′**.

Ex. 1. If $\lambda = \mu = \nu = \pi/3$, find the direction-ratios of the line joining the origin to the point (1, 2, -1). Find also the direction-cosines.

Ans. $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}};\ \frac{3}{2\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}.$

Ex. 2. Shew that the direction-ratios of a normal to the plane **XOY** are given by

$$\frac{l}{\cos\nu\cos\lambda-\cos\mu}=\frac{m}{\cos\mu\cos\nu-\cos\lambda}=\frac{n}{\sin^2\nu}=\frac{1}{\sin\nu\Delta^{\frac{1}{2}}},$$

where $\Delta\equiv 1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu.$

Ex. 3. Prove that the lines which bisect the angles **YOZ, ZOX, XOY,** internally, have direction-cosines

$$\frac{\cos\mu+\cos\nu}{2\cos\lambda/2},\quad \cos\frac{\lambda}{2},\quad \cos\frac{\lambda}{2};\ \text{etc.},$$

and that the angles between them are

$$\cos^{-1}\left(\frac{1+\cos\lambda+\cos\mu+\cos\nu}{4\cos\mu/2\cos\nu/2}\right),\ \text{etc.}$$

CHAPTER III

THE PLANE.

33. Let ABC, (fig. 23), a given plane, make intercepts OA, OB, OC on the axes, measured by a, b, c; and let ON, the normal from O to the plane, have direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and have measure p, (p is a positive number).

Equation to a plane. (i) *To find the equation to the plane* ABC *in terms of* $\cos\alpha$, $\cos\beta$, $\cos\gamma$, p.

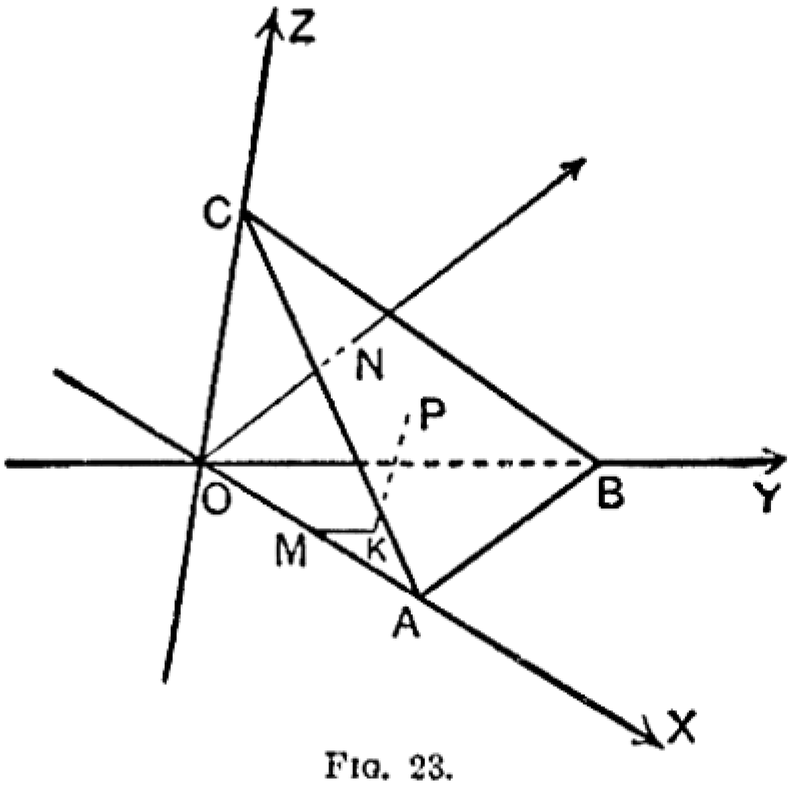

FIG. 23.

Let P, (x, y, z) be any point on the plane. Draw PK parallel to OZ to meet the plane XOY in K, and KM parallel to OY to meet OX in M. Then the measures of OM, MK, KP are x, y, z respectively, and since ON is the projection of OP on ON, and therefore equal to the sum of the projections of OM, MK, KP on ON,

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma.$$

This equation, satisfied by the coordinates of every point on the plane, represents the plane.

(ii) *To find the equation to the plane in terms of a, b, c.*

$$\mathrm{ON} = \text{projection of } \mathrm{OA} \text{ on } \mathrm{ON} = \mathrm{OA}\cos\alpha;$$

$$\therefore\ p = a\cos\alpha. \quad \text{Similarly, } b\cos\beta = c\cos\gamma = p.$$

Hence, by (i), the equation to the plane is

$$\frac{x\cos\alpha}{p} + \frac{y\cos\beta}{p} + \frac{z\cos\gamma}{p} = 1,$$

$$\text{i.e.}\quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Ex. Find the intercepts made on the coordinate axes by the plane $x + 2y - 2z = 9$. Find also the direction-cosines of the normal to the plane if the axes are rectangular. *Ans.* $9,\ 9/2,\ -9/2;\ \frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}$.

34. General equation to a plane. *The general equation of the first degree in x, y, z represents a plane.*

For $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ can be written

$$\frac{x}{-\mathrm{D}/\mathrm{A}} + \frac{y}{-\mathrm{D}/\mathrm{B}} + \frac{z}{-\mathrm{D}/\mathrm{C}} = 1,$$

and therefore represents a plane making intercepts $-\mathrm{D}/\mathrm{A}$, $-\mathrm{D}/\mathrm{B}$, $-\mathrm{D}/\mathrm{C}$ on the axes.

35. If $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ and $p = x\cos\alpha + y\cos\beta + z\cos\gamma$ represent the same plane,

$$\frac{\cos\alpha}{-\mathrm{A}} = \frac{\cos\beta}{-\mathrm{B}} = \frac{\cos\gamma}{-\mathrm{C}} = \frac{p}{\mathrm{D}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

therefore the direction-cosines of the normal to the plane $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ are proportional to A, B, C. If the axes are rectangular, each of the ratios in (1) $= \dfrac{\pm 1}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}}$.

But p is a positive number; therefore if D is positive,

$$p = \frac{\mathrm{D}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}}, \quad \cos\alpha = \frac{-\mathrm{A}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}},$$

$$\cos\beta = \frac{-\mathrm{B}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}}, \quad \text{and} \quad \cos\gamma = \frac{-\mathrm{C}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}}.$$

If D is negative, we must change the sign of $\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}$.

Cor. If the axes are rectangular, the angle between the planes

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0$$

$$\text{is} \quad \cos^{-1}\left\{\frac{\pm(aa'+bb'+cc')}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right\}.$$

Ex. 1. If the axes are rectangular, find the angle between the planes

$$\text{(i) } 2x-y+z=6, \quad x+y+2z=3\,;$$
$$\text{(ii) } 3x+4y-5z=9, \quad 2x+6y+6z=7.$$

Ans. (i) $\pi/3$, (ii) $\pi/2$.

Ex. 2. If the axes are rectangular, find the distance of the origin from the plane $6x-3y+2z-14=0$. *Ans.* 2.

Ex. 3. Shew that the equations $by+cz+d=0$, $cz+ax+d=0$, $ax+by+d=0$ represent planes parallel to **OX, OY, OZ** respectively. Find the equations to the planes through the points (2, 3, 1), (4, −5, 3) parallel to the coordinate axes.

Ans. $y+4z-7=0$, $x-z-1=0$, $4x+y-11=0$.

Ex. 4. Find the equation to the plane through (1, 2, 3) parallel to $3x+4y-5z=0$. *Ans.* $3x+4y-5z+4=0$.

Ex. 5. Prove that the equation to the plane through (α, β, γ) parallel to $ax+by+cz=0$ is $ax+by+cz=a\alpha+b\beta+c\gamma$.

Ex. 6. If the axes are rectangular and **P** is the point (2, 3, −1), find the equation to the plane through **P** at right angles to **OP**.

Ans. $2x+3y-z=14$.

Ex. 7. Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes, and find the angle between them.

Ans. $\cos^{-1} 16/21$.

Ex. 8. Prove that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

represents a pair of planes if $abc+2fgh-af^2-bg^2-ch^2=0$.

Prove that the angle between the planes is

$$\tan^{-1}\left(\frac{2(f^2+g^2+h^2-bc-ca-ab)^{\frac{1}{2}}}{a+b+c}\right).$$

Ex. 9. A variable plane is at a constant distance p from the origin and meets the axes, which are rectangular, in **A, B, C**. Through **A, B, C** planes are drawn parallel to the coordinate planes. Shew that the locus of their point of intersection is given by $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

36. Plane through three given points. The general equation to a plane contains three arbitrary constants, and therefore a plane can be found to satisfy three conditions which each involve one relation between the constants:

e.g. a plane can be found to pass through any three non-collinear points.

To find the equation to the plane through (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the equation to the plane be $ax+by+cz+d=0$.

Then
$$ax_1+by_1+cz_1+d=0,$$
$$ax_2+by_2+cz_2+d=0,$$
$$ax_3+by_3+cz_3+d=0.$$

Therefore, eliminating a, b, c, d, we obtain the required equation.

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0.$$

Ex. 1. Find the equation to the plane through the three points

$$(1, 1, 0), \quad (1, 2, 1), \quad (-2, 2, -1).$$

Ans. $2x+3y-3z=5$.

Ex. 2. Shew that the four points $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$, $(3, 3, 0)$ are coplanar.

37.[1] **Distance from a point to a plane.** *To find the distance of the point* P, (x', y', z') *from the plane*

$$p=x\cos\alpha+y\cos\beta+z\cos\gamma.$$

Suppose that p is a positive number so that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction-cosines of the normal *from* the origin *to* the plane. Change the origin to (x', y', z'), and the equation to the plane becomes

$$p=(x+x')\cos\alpha+(y+y')\cos\beta+(z+z')\cos\gamma,$$

or $$p'=x\cos\alpha+y\cos\beta+z\cos\gamma,$$

where $$p'\equiv p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

Hence the distance of (x', y', z'), the new origin, from the plane is $$p'\equiv p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

If P is on the same side of the plane as the original origin O, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are still the direction-cosines of the normal *from* the new origin, P, *to* the plane, and therefore p' or $p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$ is positive. If P and O are on opposite sides of the plane, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction-cosines of the normal *from* the plane *to* P, and therefore p' or $p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$ is negative. Hence, if p is positive,

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[1] See Appendix, p. i.

$$p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$$

is positive if (x', y', z') is any point on the same side of the plane as the origin, and negative if (x', y', z') is any point on the side of the plane remote from the origin.

Cor. 1. The distance of (x', y', z') from the plane

$$ax+by+cz+d=0,$$

if the axes are rectangular, is given by $\dfrac{ax'+by'+cz'+d}{\pm\sqrt{a^2+b^2+c^2}}$.

If d is positive the positive sign is to be taken, as it gives a positive value for the perpendicular from the origin.

Cor. 2. If d is positive, the expression $ax'+by'+cz'+d$ is positive if (x', y', z') and the origin are on the same side of the plane $ax+by+cz+d=0$, and negative if they are on opposite sides.

Ex. 1. If P is (x', y', z'), shew that the projection of **OP** on the normal to the plane

$$p=x\cos\alpha+y\cos\beta+z\cos\gamma \text{ is } x'\cos\alpha+y'\cos\beta+z'\cos\gamma,$$

and deduce the results of § 37.

Ex. 2. Find the distances of the points $(2, 3, -5)$, $(3, 4, 7)$ from the plane $x+2y-2z=9$. Are the points on the same side of the plane?

Ans. 3, 4, No.

Ex. 3. Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x+3y-6z=2$.

Ans. $3x^2+8y^2+35z^2-36yz-24zx+12xy-8x-12y+24z+4=0$.

Ex. 4. Find the locus of a point the sum of the squares of whose distances from the planes $x+y+z=0$, $x-z=0$, $x-2y+z=0$, is 9.

Ans. $x^2+y^2+z^2=9$.

Ex. 5. The sum of the squares of the distances of a point from the planes $x+y+z=0$, $x-2y+z=0$ is equal to the square of its distance from the plane $x=z$. Prove that the equation to the locus of the point is $y^2+2xz=0$. By turning the axes of x and z in their plane through angles of 45°, prove that the locus is a right circular cone whose semi-vertical angle is 45°.

38. Planes bisecting the angles between given planes.

To find the planes bisecting the angles between the given planes $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$, *the axes being rectangular.*

We can always write the equations so that d and d' are positive. Then the equation

$$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}=\frac{a'x+b'y+c'z+d'}{\sqrt{a'^2+b'^2+c'^2}}$$

represents the locus of points equidistant from the given planes, and since the expressions

$$ax+by+cz+d,\quad a'x+b'y+c'z+d'$$

in the equation have the same sign, the points are on the origin side of both planes or on the non-origin side of both. The locus is therefore the plane bisecting that angle between the given planes which contains the origin. Similarly,

$$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}=-\frac{a'x+b'y+c'z+d'}{\sqrt{a'^2+b'^2+c'^2}}$$

represents the plane bisecting the other angle between the given planes.

Ex. 1. Shew that the origin lies in the acute angle between the planes $x+2y+2z=9$, $4x-3y+12z+13=0$. Find the planes bisecting the angles between them, and point out which bisects the acute angle.

Ans. Acute, $25x+17y+62z-78=0$; obtuse, $x+35y-10z-156=0$.

***Ex. 2.** Shew that the plane $ax+by+cz+d=0$ divides the join of (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio

$$-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

[The point $\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$ lies on the plane if

$$\lambda(ax_2+by_2+cz_2+d)+ax_1+by_1+cz_1+d=0.]$$

***Ex. 3.** Hence shew that the planes $u\equiv ax+by+cz+d=0$, $v\equiv a'x+b'y+c'z+d'=0$, $u+\lambda v=0$, $u-\lambda v=0$ divide any transversal harmonically.

Let P, (x_1, y_1, z_1) be on the plane $u=0$, then $u_1\equiv ax_1+by_1+cz_1+d=0$. Let Q, (x_2, y_2, z_2) be on the plane $v=0$, then $v_2\equiv a'x_2+b'y_2+c'z_2+d'=0$. The planes $u\pm\lambda v=0$ divide PQ in the ratios

$$-\frac{u_1\pm\lambda v_1}{u_2\pm\lambda v_2},\ i.e.\ \mp\frac{\lambda v_1}{u_2},$$

i.e. divide PQ harmonically.

***Ex. 4.** If A, P, B, Q are any four collinear points, the anharmonic ratio, or cross-ratio of the range APBQ, is defined to be

$$\frac{AP}{PB} \div \frac{AQ}{QB} \text{ or } \frac{AP.QB}{AQ.PB}.$$

Prove that four given planes that pass through one line cut any transversal in a range of constant cross-ratio.

If $u=0$, $v=0$ are two planes through the line, the equations to the four given planes can be written, $u+\lambda_r v=0$, $r=1, 2, 3, 4$. Let A, B, (x_1, y_1, z_1), (x_2, y_2, z_2) lie on the planes $u+\lambda_1 v=0$, $u+\lambda_3 v=0$ respectively. Then $u_1+\lambda_1 v_1=0$ and $u_2+\lambda_3 v_2=0$. If P, Q lie on the planes $u+\lambda_2 v=0$, $u+\lambda_4 v=0$, then by Ex. 2,

$$\frac{AP}{PB} = -\frac{u_1+\lambda_2 v_1}{u_2+\lambda_2 v_2}, \quad \frac{AQ}{QB} = -\frac{u_1+\lambda_4 v_1}{u_2+\lambda_4 v_2},$$

and therefore

$$\frac{AP.QB}{AQ.PB} = \frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_4)}.$$

This constant cross-ratio is called the cross-ratio of the four planes.

***Ex. 5.** P, Q, R, S are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral. Prove that

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1.$$

THE STRAIGHT LINE.

39. The equations to a line. Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the coordinates of any point on the line of intersection of the planes which they represent, and therefore the two equations together represent that line. Thus $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ represent a straight line.

40. Symmetrical form of equations. The equations to a straight line can be found in a more symmetrical form. If the line passes through a given point P, (x', y', z') and has direction-ratios l, m, n,

$$l=\frac{x-x'}{r}, \quad m=\frac{y-y'}{r}, \quad n=\frac{z-z'}{r},$$

where Q, (x, y, z) is any point on it, and the measure of PQ is r, (§ 21, Cor. 3; § 29, Cor. 3). And therefore the coordinates of any point on the line satisfy the equations

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n} \quad (=r).$$

These equations enable us to express the coordinates of a variable point on the line in terms of one parameter r, for

$$x=x'+lr, \quad y=y'+mr, \quad z=z'+nr.$$

Conversely, any equations of the form

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

represent a straight line passing through the point (a, b, c) and having direction-ratios proportional to l, m, n.

Ex. 1. Find where the line $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}$ meets the plane $2x+4y-z+1=0$. *Ans.* $(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3})$.

Ex. 2. Find the points in which the line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}$ cuts the surface $11x^2-5y^2+z^2=0$. *Ans.* $(1, 2, 3)$, $(2, -3, 1)$.

Ex. 3. If the axes are rectangular, find the distance from the point $(3, 4, 5)$ to the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=2$. *Ans.* -6.

Ex. 4. Find the distance of the point $(1, -2, 3)$ from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$, (rectangular axes). *Ans.* 1.

Ex. 5. Shew that if the axes are rectangular, the equations to the perpendicular from the point (α, β, γ) to the plane $ax+by+cz+d=0$ are $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$, and deduce the perpendicular distance of the point (α, β, γ) from the plane.

Ex. 6. If the axes are rectangular, the equations to the line through (α, β, γ) at right angles to the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$, $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ are

$$\frac{x-\alpha}{m_1n_2-m_2n_1}=\frac{y-\beta}{n_1l_2-n_2l_1}=\frac{z-\gamma}{l_1m_2-l_2m_1}.$$

Ex. 7. If the axes are rectangular, and if l_1, m_1, n_1; l_2, m_2, n_2 are direction-cosines, shew that the equations to the planes through the lines which bisect the angles between

$$x/l_1=y/m_1=z/n_1 \quad \text{and} \quad x/l_2=y/m_2=z/n_2,$$

and at right angles to the plane containing them, are

$$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

Ex. 8. A line through the origin makes angles α, β, γ with its projections on the coordinate planes, which are rectangular. The distances of any point (x, y, z) from the line and its projections are d, a, b, c. Prove that

$$d^2=(a^2-x^2)\cos^2\alpha+(b^2-y^2)\cos^2\beta+(c^2-z^2)\cos^2\gamma.$$

41. Line through two points. If P, Q are (x_1, y_1, z_1), (x_2, y_2, z_2), the direction-ratios of PQ are proportional to x_2-x_1, y_2-y_1, z_2-z_1, and therefore the equations to PQ are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

By § 8, the coordinates of a variable point of the line in terms of one parameter, λ, are

$$x=\frac{\lambda x_2+x_1}{\lambda+1},\quad y=\frac{\lambda y_2+y_1}{\lambda+1},\quad z=\frac{\lambda z_2+z_1}{\lambda+1}.$$

Ex. 1. Find the point where the line joining (2, 1, 3), (4, −2, 5) cuts the plane $2x+y-z=3$. *Ans.* (0, 4, 1).

Ex. 2. Prove that the line joining the points (4, −5, −2), (−1, 5, 3) meets the surface $2x^2+3y^2-4z^2=1$ in coincident points.

42. Direction-ratios from equations. The planes through the origin parallel to

$$ax+by+cz+d=0,\quad a'x+b'y+c'z+d'=0$$

are given by

$$ax+by+cz=0,\quad a'x+b'y+c'z=0.$$

Hence the equations

$$ax+by+cz=0=a'x+b'y+c'z$$

together represent the straight line through the origin parallel to the line given by

$$ax+by+cz+d=0=a'x+b'y+c'z+d'.$$

They may be written

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b},$$

and therefore the direction-ratios of the two lines are proportional to $bc'-b'c$, $ca'-c'a$, $ab'-a'b$. Again the second line meets the plane $z=0$ in the point

$$\left(\frac{bd'-b'd}{ab'-a'b},\ \frac{da'-d'a}{ab'-a'b},\ 0\right);$$

therefore the equations to the second line in the symmetrical form are

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

Ex. 1. The equations to a line through (a, b, c) parallel to the plane **XOY** are

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{0}, \quad\text{..........................(1)}$$

since the direction-ratios are $l, m, 0$. Again the line lies in the plane $z=c$, and therefore its equations can be written

$$m(x-a)=l(y-b),\ z=c, \quad\text{..........................(2)}$$

and (1) is to be considered the symmetrical form of (2).

Ex. 2. Find the equations to the line joining $(2, 4, 3)$, $(-3, 5, 3)$. The equations are $\frac{x-2}{-5}=\frac{y-4}{1}=\frac{z-3}{0}$. Therefore the line is parallel to the plane **XOY**, as is evident, since the z-coordinates of two points on it are equal to 3. The equations can also be written

$$x+5y=22,\ z=3.$$

Ex. 3. The equations to the straight line through (a, b, c) parallel to **OZ** are $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$ or $x=a$, $y=b$.

Ex. 4. Prove that the equations to the line of intersection of the planes $4x+4y-5z=12$, $8x+12y-13z=32$ can be written

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}.$$

Ex. 5. Shew that the line $2x+2y-z-6=0=2x+3y-z-8$ is parallel to the plane $y=0$, and find the coordinates of the point where it meets the plane $x=0$. *Ans.* $(0, 2, -2)$

Ex. 6. Prove that the lines

$$2x+3y-4z=0=3x-4y+z,\quad 5x-y-3z+12=0=x-7y+5z-6$$

are parallel.

Ex. 7. Find the angle between the lines

$$x-2y+z=0=x+y-z,\quad x+2y+z=0=8x+12y+5z,$$

(rectangular axes). *Ans.* $\cos^{-1}8/\sqrt{406}$.

Ex. 8. Find the equations to the line through the point $(1, 2, 3)$ parallel to the line $x-y+2z=5$, $3x+y+z=6$.

Ans. $\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}$.

43.[1] Constants in the equations to a line. The equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

may be written

$$\left.\begin{aligned} x&=\frac{l}{m}y+a-\frac{lb}{m} \\ y&=\frac{m}{n}z+b-\frac{mc}{n} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

which are of the form

$$\left.\begin{aligned} x&=\mathsf{A}y+\mathsf{B} \\ y&=\mathsf{C}z+\mathsf{D} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and therefore the general equations to a straight line contain four arbitrary constants. The equations (1) represent the planes passing through the line and parallel to OZ and OX respectively, and by a choice of such planes to define any given line its equations can be put in the form (2), which is the form with the smallest possible number of arbitrary constants.

Ex. 1. Prove that the symmetrical form of the equations to the line given by $x=ay+b$, $z=cy+d$ is $\dfrac{x-b}{a}=\dfrac{y}{1}=\dfrac{z-d}{c}$.

Ex. 2. Prove that the lines

$$x=ay+b, \quad z=cy+d, \quad x=a'y+b', \quad z=c'y+d',$$

are perpendicular if $aa'+cc'+1=0$.

Ex. 3. Find a, b, c, d, so that the line $x=ay+b$, $z=cy+d$ may pass through the points $(3, 2, -4)$, $(5, 4, -6)$, and hence shew that the given points and $(9, 8, -10)$ are collinear.

Ans. $a=1$, $b=1$, $c=-1$, $d=-2$.

Ex. 4. Prove that the line $x=pz+q$, $y=rz+s$, intersects the conic $z=0$, $ax^2+by^2=1$, if $aq^2+bs^2=1$.

Hence shew that the coordinates of any point on a line which intersects the conic and passes through the point (α, β, γ) satisfy the equation $a(\gamma x-\alpha z)^2+b(\gamma y-\beta z)^2=(z-\gamma)^2$.

Ex. 5. Prove that a line which passes through the point (α, β, γ) and intersects the parabola $y=0$, $z^2=4ax$, lies on the surface

$$(\beta z-\gamma y)^2=4a(\beta-y)(\beta x-\alpha y).$$

Ex. 6. Find the equations to the planes through the lines

(i) $\dfrac{x-2}{2}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$, (ii) $2x+3y-5z-4=0=3x-4y+5z-6$,

parallel to the coordinate axes.

Ans. (i) $5y-4z+1=0$, $2z-5x+2=0$, $2x-y-1=0$;

(ii) $17y-25z=0$, $5z-17x+34=0$, $5x-y-10=0$.

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[1] See Appendix, p. i.

***Ex. 7.** If the axes are oblique the distance of the point (x', y', z') from the plane $ax+by+cz+d=0$ is given by

$$\frac{\pm(ax'+by'+cz'+d)(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}{\{\Sigma a^2\sin^2\lambda-2\Sigma bc(\cos\lambda-\cos\mu\cos\nu)\}^{\frac{1}{2}}}.$$

***Ex. 8.** The distance of (x', y', z') from the line $x/a=y/b=z/c$ is given by

$$d^2=\frac{\Sigma(bz-cy)^2\sin^2\lambda+2\Sigma(cx-az)(ay-bx)(\cos\mu\cos\nu-\cos\lambda)}{a^2+b^2+c^2+2bc\cos\lambda+2ca\cos\mu+2ab\cos\nu}.$$

***Ex. 9.** Prove that the direction-cosines of the normal to the plane **OXY** are $0, 0, \dfrac{\Delta^{\frac{1}{2}}}{\sin\nu}$,

where $$\Delta\equiv 1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu.$$

If the angles that **OX, OY, OZ** make with the planes **YOZ, ZOX, XOY** are α, β, γ, prove that

$$\frac{\sin\alpha}{\operatorname{cosec}\lambda}=\frac{\sin\beta}{\operatorname{cosec}\mu}=\frac{\sin\gamma}{\operatorname{cosec}\nu}=\Delta^{\frac{1}{2}}.$$

If the angles between the planes **ZOX, XOY**, etc., are **A, B, C**, prove that

$$\text{(i)}\ \cos\lambda-\cos\mu\cos\nu=\sin\mu\sin\nu\cos \mathsf{A},$$

$$\text{(ii)}\ \frac{\sin \mathsf{A}}{\sin\lambda}=\frac{\sin \mathsf{B}}{\sin\mu}=\frac{\sin \mathsf{C}}{\sin\nu}.$$

44. The plane and the straight line. Let the equations $ax+by+cz+d=0$, $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ represent a given plane and straight line. Their point of intersection is

$$(\alpha+lr,\ \beta+mr,\ \gamma+nr),$$

where r is given by

$$r(al+bm+cn)+a\alpha+b\beta+c\gamma+d=0.$$

But r is proportional to the distance of the point from (α, β, γ). Therefore the line is parallel to the plane if

$$al+bm+cn=0 \text{ and } a\alpha+b\beta+c\gamma+d\neq 0.$$

If the axes are rectangular, the direction-cosines of the normal to the plane and of the line are proportional to a, b, c; l, m, n; and therefore if the line is normal to the plane,

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}.$$

Cor. The conditions that the line should lie in the plane are

$$al+bm+cn=0$$

and

$$a\alpha+b\beta+c\gamma+d=0.$$

Ex. 1. Prove that the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ is parallel to the plane $4x+4y-5z=0$.

Ex. 2. Prove that the planes $2x-3y-7z=0$, $3x-14y-13z=0$, $8x-31y-33z=0$ pass through one line.

Ex. 3. Find the equation to the plane through $(2, -3, 1)$ normal to the line joining $(3, 4, -1)$, $(2, -1, 5)$, (axes rectangular).

Ans. $x+5y-6z+19=0$.

Ex. 4. Find the equation to the plane through the points $(2, -1, 0)$, $(3, -4, 5)$ parallel to the line $2x=3y=4z$.

Ans. $29x-27y-22z=85$.

Ex. 5. Prove that the join of $(2, 3, 4)$, $(3, 4, 5)$ is normal to the plane through $(-2, -3, 6)$, $(4, 0, -3)$, $(0, -1, 2)$, the axes being rectangular.

Ex. 6. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$, (rectangular axes). *Ans.* 13.

Ex. 7. Find the equations to the planes through the point $(-1, 0, 1)$ and the lines

$$4x-3y+1=0=y-4z+13\ ; \quad 2x-y-2=0=z-5,$$

and shew that the equations to the line through the given point which intersects the two given lines can be written

$$x=y-1=z-2.$$

Ex. 8. Find the equation to the plane through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

parallel to the line $$\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}.$$

Ans. $\Sigma(x-\alpha)(mn'-m'n)=0$.

Ex. 9. The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation to the plane in its new position is

$$lx+my\pm z\sqrt{l^2+m^2}\tan\alpha=0.$$

Ex. 10. Find the equations to the line through (f, g, h) which is parallel to the plane $lx+my+nz=0$ and intersects the line

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0.$$

Ans. $$\begin{cases} l(x-f)+m(y-g)+n(z-h)=0, \\ \dfrac{ax+by+cz+d}{af+bg+ch+d}=\dfrac{a'x+b'y+c'z+d'}{a'f+b'g+c'h+d'}. \end{cases}$$

Ex. 11. The axes being rectangular, find the equations to the perpendicular from the origin to the line

$$x+2y+3z+4=0, \quad 2x+3y+4z+5=0.$$

Find also the coordinates of the foot of the perpendicular.

(The perpendicular is the line of intersection of the plane through the origin and the line and the plane through the origin perpendicular to the line.)

Ans. $\dfrac{x}{2}=\dfrac{y}{-1}=\dfrac{z}{-4}$; $\left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{-4}{3}\right)$.

Ex. 12. The equations to **AB** referred to rectangular axes are $\dfrac{x}{2}=\dfrac{y}{-3}=\dfrac{z}{6}$. Through a point **P**, (1, 2, 5) **PN** is drawn perpendicular to **AB**, and **PQ** is drawn parallel to the plane $3x+4y+5z=0$ to meet **AB** in **Q**. Find the equations to **PN** and **PQ** and the coordinates of **N** and **Q**.

Ans. $\dfrac{x-1}{-3}=\dfrac{y-2}{176}=\dfrac{z-5}{89}$; $\dfrac{x-1}{4}=\dfrac{y-2}{-13}=\dfrac{z-5}{8}$;

$\left(\dfrac{52}{49}, \dfrac{-78}{49}, \dfrac{156}{49}\right)$; $\left(3, \dfrac{-9}{2}, 9\right)$.

Ex. 13. Through a point **P**, (x', y', z') a plane is drawn at right angles to **OP** to meet the axes (rectangular) in **A**, **B**, **C**. Prove that the area of the triangle **ABC** is $\dfrac{r^5}{2x'y'z'}$, where r is the measure of **OP**.

Ex. 14. The axes are rectangular and the plane $x/a+y/b+z/c=1$ meets them in **A**, **B**, **C**. Prove that the equations to **BC** are $\dfrac{x}{0}=\dfrac{y}{b}=\dfrac{z-c}{-c}$; that the equation to the plane through **OX** at right angles to **BC** is $by=cz$; that the three planes through **OX**, **OY**, **OZ**, at right angles to **BC**, **CA**, **AB** respectively, pass through the line $ax=by=cz$; and that the coordinates of the orthocentre of the triangle **ABC** are :

$$\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}}, \quad \frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}}, \quad \frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}.$$

Ex. 15. If the axes are rectangular, the distance of the point (x_0, y_0, z_0) from the line

$$u \equiv ax+by+cz+d=0, \quad v \equiv a'x+b'y+c'z+d'=0$$

is given by $$\left\{\frac{(a'u_0-av_0)^2+(b'u_0-bv_0)^2+(c'u_0-cv_0)^2}{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}\right\}^{\frac{1}{2}},$$

where $u_0 \equiv ax_0+by_0+cz_0+d$, and $v_0 \equiv a'x_0+b'y_0+c'z_0+d'$.

Ex. 16. Find the equation to the plane through the line

$$u \equiv ax+by+cz+d=0, \quad v \equiv a'x+b'y+c'z+d'=0,$$

parallel to the line $x/l=y/m=z/n$.

Ans. $u(a'l+b'm+c'n)=v(al+bm+cn)$.

Ex. 17. Find the equation to the plane through the lines

$$ax+by+cz=0=a'x+b'y+c'z, \quad \alpha x+\beta y+\gamma z=0=\alpha'x+\beta'y+\gamma'z.$$

Ans.
$$\begin{vmatrix} x, & y, & z \\ bc'-b'c, & ca'-c'a, & ab'-a'b \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.$$

Ex. 18. Prove that the plane through the point (α, β, γ) and the line $x=py+q=rz+s$ is given by

$$\begin{vmatrix} x, & py+q, & rz+s \\ \alpha, & p\beta+q, & r\gamma+s \\ 1, & 1, & 1 \end{vmatrix}=0.$$

Ex. 19. The distance of the point (ξ, η, ζ) from the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, measured parallel to the plane $ax+by+cz=0$, is given by

$$d^2=\frac{(a^2+b^2+c^2)\Sigma\{m(\gamma-\zeta)-n(\beta-\eta)\}^2-\{\Sigma(\alpha-\xi)(bn-cm)\}^2}{(al+bm+cn)^2}.$$

Deduce the perpendicular distance of the point from the line.

***Ex. 20.** If the axes are oblique, the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is normal to the plane $ax+by+cz+d=0$, if

$$\frac{\frac{\partial\phi}{\partial l}}{a}=\frac{\frac{\partial\phi}{\partial m}}{b}=\frac{\frac{\partial\phi}{\partial n}}{c}. \quad \text{(See § 31.)}$$

***Ex. 21.** Shew that the equation to the plane through **OZ** at right angles to the plane **XOY** is

$$x(\cos\mu\cos\nu-\cos\lambda)=y(\cos\nu\cos\lambda-\cos\mu).$$

***Ex. 22.** Shew that the planes through **OX, OY, OZ**, at right angles to the planes **YOZ, ZOX, XOY**, pass through the line

$$x(\cos\mu\cos\nu-\cos\lambda)=y(\cos\nu\cos\lambda-\cos\mu)=z(\cos\lambda\cos\mu-\cos\nu).$$

***Ex. 23.** The planes through **O** normal to **OX, OY, OZ** cut the planes **YOZ, ZOX, XOY** in lines which lie in the plane

$$\frac{x}{\cos\lambda}+\frac{y}{\cos\mu}+\frac{z}{\cos\nu}=0.$$

***Ex. 24.** Shew that the line in Ex. 22 is at right angles to the plane in Ex. 23.

***Ex. 25.** If **P** is the point (x', y', z') and the perpendiculars from **P** to the coordinate planes are p_1, p_2, p_3, prove that

$$\frac{p_1\sin\lambda}{x'}=\frac{p_2\sin\mu}{y'}=\frac{p_3\sin\nu}{z'}=\Delta^{\frac{1}{2}}.$$

Deduce that the planes bisecting the interior angles between the coordinate planes pass through the line

$$\frac{x}{\sin\lambda}=\frac{y}{\sin\mu}=\frac{z}{\sin\nu}.$$

***Ex. 26.** Shew that the squares of the distances of P, (x', y', z') from the coordinate axes are

$$y'^2 \sin^2\nu + z'^2 \sin^2\mu + 2y'z'(\cos\lambda - \cos\mu\cos\nu), \text{ etc.}$$

***Ex. 27.** Prove that the equation to the plane through O normal to

$$\frac{x}{\sin\lambda} = \frac{y}{\sin\mu} = \frac{z}{\sin\nu}$$

is
$$x\cos\frac{-\lambda+\mu+\nu}{2} + y\cos\frac{\lambda-\mu+\nu}{2} + z\cos\frac{\lambda+\mu-\nu}{2} = 0.$$

45. The intersection of three planes. Before proceeding to the general discussion of the intersection of three given planes we will consider three typical numerical cases.

Solving the equations

$$\begin{aligned} 3x - 4y + 5z &= 10, \\ 2x - y + z &= 3, \\ x - 3y + 2z &= 1, \end{aligned}$$

we obtain $x=1$, $y=2$, $z=3$, and hence the three planes represented by the given equations pass through the point (1, 2, 3).

Let us now attempt to solve the equations

$$\begin{aligned} \text{(i)}\quad 2x - 4y + 2z &= 5, \\ \text{(ii)}\quad 5x - y - z &= 8, \\ \text{(iii)}\quad x + y - z &= 7. \end{aligned}$$

Eliminate z from (ii) and (iii), then from (i) and (ii), and we get
$$4x - 2y = 1, \quad 4x - 2y = 7.$$

Whence subtracting, $0 \,.\, x + 0 \,.\, y = 6.$

Similarly, eliminating y from (i) and (ii), then from (ii) and (iii), we get
$$6x - 2z = 9, \quad 6x - 2z = 15,$$

whence
$$0 \,.\, x + 0 \,.\, z = 6.$$

There are, therefore, no finite values of x, y, z, which satisfy all the given equations. The equations $0 \,.\, x + 0 \,.\, y = 6$, $0 \,.\, x + 0 \,.\, z = 6$, are limiting forms of $\frac{x}{k} + \frac{y}{k} = 6$, $\frac{x}{k} + \frac{z}{k} = 6$, as k tends to infinity, and hence we may say that any point whose coordinates satisfy the three given equations is at an infinite distance. We easily find that the lines of intersection of any two of the planes are parallel to the line

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3},$$

and it is evident that no two of the planes are parallel, so that the three planes form a triangular prism. Thus if we are given the three equations to the faces of a triangular prism, and we attempt to solve them, we obtain a paradoxical equation of the form $k=0$, where k is a number different from zero.

Consider, in the third place, the equations

$$\begin{aligned} &\text{(i)} & 12x-\ y+2z&=35,\\ &\text{(ii)} & 3x+\ y+\ z&=\ 7,\\ &\text{(iii)} & x+2y+\ z&=\ 0. \end{aligned}$$

Eliminating z between (i) and (ii), and then between (ii) and (iii), we obtain

$$6x-3y=21,\quad 2x-y=7.$$

Similarly, if we eliminate x in any way between the equations, we get $\quad 5y+2z+7=0.$

Thus all points whose coordinates satisfy the given equations lie upon both of the planes $2x-y=7$, $5y+2z+7=0$, or the common points of the three planes lie upon a straight line, that is, the three planes intersect in a straight line.

Ex. 1. Examine the nature of the intersection of the sets of planes:

(i) $2x-5y+z=3,\quad x+y+4z=5,\quad x+3y+6z=1$;

(ii) $3x+4y+6z=5,\quad 6x+5y+9z=10,\quad 3x+3y+5z=5$;

(iii) $x+y+z=6,\quad 2x+3y+4z=20,\quad x-y+z=2$;

(iv) $x+2y+3z=6,\quad 3x+4y+5z=2,\quad 5x+4y+3z+18=0$;

(v) $2x+3y+4z=6,\quad 3x+4y+5z=20,\quad x+2y+3z=2$;

(vi) $2x-y+z=4,\quad 5x+7y+2z=0,\quad 3x+4y-2z+3=0$;

(vii) $3x-y+z=5,\quad 2x+4y+z+10=0,\quad 6x-2y+2z+9=0.$

Ans. (i) Planes form prism; (ii) planes pass through line

$$\frac{3x-5}{2}=\frac{y}{1}=\frac{z}{-1};$$

(iii) planes intersect at (1, 2, 3); (iv) planes pass through line

$$\frac{x+10}{1}=\frac{y-8}{-2}=\frac{z}{1};$$

(v) planes form prism; (vi) planes intersect at (1, −1, 1); (vii) two planes parallel, third intersects them.

Ex. 2. Prove that the three planes $2x+y+z=3$, $x-y+2z=4$, $x+z=2$, form a triangular prism, and find the area of a normal section of the prism. *Ans.* $\sqrt{3}/18$.

We shall now consider the general case.

Let the equations to the planes be

$$u_1 \equiv a_1x+b_1y+c_1z+d_1=0, \quad \ldots\ldots\ldots\ldots\ldots(1)$$
$$u_2 \equiv a_2x+b_2y+c_2z+d_2=0, \quad \ldots\ldots\ldots\ldots\ldots(2)$$
$$u_3 \equiv a_3x+b_3y+c_3z+d_3=0. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Solving the equations (1), (2), (3), we obtain:

$$\frac{x}{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}},$$

which, by contracting the denominators, we may write,

$$\frac{x}{|b_1,\ c_2,\ d_3|} = \frac{-y}{|a_1,\ c_2,\ d_3|} = \frac{z}{|a_1,\ b_2,\ d_3|} = \frac{-1}{|a_1,\ b_2,\ c_3|}. \quad \ldots(4)$$

Let $\Delta \equiv |a_1,\ b_2,\ c_3|$, $\mathsf{A}_1 \equiv \dfrac{\partial\Delta}{\partial a_1} \equiv (b_2c_3-b_3c_2)$,

$$\mathsf{B}_1 \equiv \frac{\partial\Delta}{\partial b_1} \equiv (c_2a_3-c_3a_2), \quad \mathsf{C}_1 \equiv \frac{\partial\Delta}{\partial c_1} \equiv (a_2b_3-a_3b_2);\ \text{etc.}$$

Then

$$\mathsf{B}_2\mathsf{C}_3-\mathsf{B}_3\mathsf{C}_2=a_1\Delta, \quad \mathsf{C}_2\mathsf{A}_3-\mathsf{C}_3\mathsf{A}_2=b_1\Delta, \quad \mathsf{A}_2\mathsf{B}_3-\mathsf{A}_3\mathsf{B}_2=c_1\Delta;\ \text{etc.}$$

Therefore, if $\Delta=0$,

$$\frac{\mathsf{A}_2}{\mathsf{A}_3}=\frac{\mathsf{B}_2}{\mathsf{B}_3}=\frac{\mathsf{C}_2}{\mathsf{C}_3} \text{ and } \frac{\mathsf{A}_3}{\mathsf{A}_1}=\frac{\mathsf{B}_3}{\mathsf{B}_1}=\frac{\mathsf{C}_3}{\mathsf{C}_1}. \quad \ldots\ldots\ldots\ldots(5)$$

If $\Delta\neq0$, the equations (4) give finite values of x, y, z, and therefore the three given planes have a point of intersection at a finite distance.

But if $\Delta=0$ and $|b_1,\ c_2,\ d_3|\neq0$, the given equations are not satisfied by any finite value of x. Since

$$|b_1,\ c_2,\ d_3| \equiv d_1\mathsf{A}_1+d_2\mathsf{A}_2+d_3\mathsf{A}_3,$$

A_1, A_2, A_3 cannot all be zero, and therefore the three planes are not all parallel. Again, the lines of intersection of the planes are parallel to

$$\frac{x}{\mathsf{A}_1}=\frac{y}{\mathsf{B}_1}=\frac{z}{\mathsf{C}_1}, \quad \frac{x}{\mathsf{A}_2}=\frac{y}{\mathsf{B}_2}=\frac{z}{\mathsf{C}_2}, \quad \frac{x}{\mathsf{A}_3}=\frac{y}{\mathsf{B}_3}=\frac{z}{\mathsf{C}_3},$$

and hence, by (5), are parallel. If A_1, A_2, A_3 are all different from zero, no two of the given planes are parallel, and the planes form a triangular prism. The edges are parallel to XOY if $\mathsf{C}_1=\mathsf{C}_2=\mathsf{C}_3=0$, (and so $|\,a_1\ b_2\ d_3\,|=0$), or to OX if B_1, B_2, B_3, C_1, C_2, C_3, are all zero (and so $|\,a_1\ c_2\ d_3\,|=0$ and $|\,a_1\ b_2\ d_3\,|=0$). If $\mathsf{A}_1=\mathsf{B}_1=\mathsf{C}_1=0$, and $\mathsf{A}_2, \mathsf{A}_3\neq 0$, the planes u_2 and u_3 are parallel and u_1 meets them, and we have a limiting case of a triangular prism when one of the edges is at an infinite distance. Since d_2 and d_3 are not both zero, we have $|\,b_1\ c_2\ d_3\,|\neq 0$. Hence for a triangular prism, $\Delta=0$ and one of the other three determinants is different from zero.

It is to be noted that in this case

$$\mathsf{A}_1u_1+\mathsf{A}_2u_2+\mathsf{A}_3u_3\equiv|\,b_1,\ c_2,\ d_3\,|\neq 0,$$

that is, when three planes are parallel to one line their equations can be combined so as to form a paradoxical equation $k=0$, where k is a quantity different from zero. Conversely, if three numbers l, m, n can be found so that

$$lu_1+mu_2+nu_3\equiv k,$$

where k is independent of x, y, z, and is not zero, then the three planes are parallel to one line, and if no two of them are parallel, form a triangular prism. For

$$a_1l+a_2m+a_3n=0,\quad b_1l+b_2m+b_3n=0,$$
$$c_1l+c_2m+c_3n=0,\quad d_1l+d_2m+d_3n\neq 0.$$

Therefore $|\,a_1,\ b_2,\ c_3\,|=0$ and $|\,b_1,\ c_2,\ d_3\,|\neq 0$.

Suppose now that $\Delta=0$, $|\,b_1,\ c_2,\ d_3\,|=0$ and $\mathsf{A}_1\neq 0$, (A_1 is one of the common minors of Δ and $|\,b_1,\ c_2,\ d_3\,|$). As in the last case, the three planes are parallel to one line. But since $|\,b_1,\ c_2,\ d_3\,|=0$, the three lines in which the planes cut the plane YOZ, viz.,

$$x=0,\quad b_1y+c_1z+d_1=0;$$
$$x=0,\quad b_2y+c_2z+d_2=0;$$
$$x=0,\quad b_3y+c_3z+d_3=0$$

are concurrent. Their common point is given by $x=0$,

$$\frac{y}{c_2d_3-c_3d_2}=\frac{z}{d_2b_3-d_3b_2}=\frac{1}{\mathsf{A}_1},$$

and since $\mathsf{A}_1\neq 0$, it is at a finite distance. Hence, since the three planes are parallel to one line and pass through a point in the plane **YOZ**, they pass through one line.

It follows now that $|a_1,\ c_2,\ d_3|$ and $|a_1,\ b_2,\ d_3|$, the remaining two determinants in (4), are zero. For since the planes pass through one line, their lines of intersection with the plane **ZOX**, viz.,

$$y=0,\ \ a_1x+c_1z+d_1=0;\quad y=0,\ \ a_2x+c_2z+d_2=0;$$
$$y=0,\ \ a_3x+c_3z+d_3=0$$

are concurrent. Therefore $|a_1,\ c_2,\ d_3|=0$, and similarly, $|a_1,\ b_2,\ d_3|=0$.

Again, if $|a_1,\ b_2,\ d_3|=0$, $|a_1,\ c_2,\ d_3|=0$ and $a_2d_3-a_3d_2$, (any one of the common minors), is not zero, the lines of intersection of the given planes with the planes **ZOX** and **XOY** are concurrent. The points of concurrence are given by

$$y=0,\ \ \frac{x}{c_2d_3-c_3d_2}=\frac{z}{d_2a_3-d_3a_2}=\frac{1}{a_2c_3-a_3c_2};$$

$$z=0,\ \ \frac{x}{b_2d_3-b_3d_2}=\frac{y}{d_2a_3-d_3a_2}=\frac{1}{a_2b_3-b_3a_2};$$

and since $d_2a_3-d_3a_2\neq 0$, they are not coincident. The planes have therefore two common points and thus pass through one line. It follows then that $|a_1,\ b_2,\ c_3|$ and $|b_1,\ c_2,\ d_3|$ are both zero.

If, therefore, any two of the determinants

$$|b_1,\ c_2,\ d_3|,\quad |a_1,\ c_2,\ d_3|,\quad |a_1,\ b_2,\ d_3|,\quad |a_1,\ b_2,\ c_3|$$

are zero, and one of their common minors is not zero, the remaining two determinants are zero,* and the three planes have a line of intersection at a finite distance.

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* This is easily proved algebraically. If $\Delta=0$, $|b_1,\ c_2,\ d_3|=0$, and $A_1\neq 0$, then, since

$$\Delta\equiv a_1A_1+a_2A_2+a_3A_3=0,$$

and

$$|b_1,\ c_2,\ d_3|\equiv d_1A_1+d_2A_2+d_3A_3=0,$$

we have

$$\frac{A_1}{a_2d_3-a_3d_2}=\frac{A_2}{a_3d_1-a_1d_3}=\frac{A_3}{a_1d_2-a_2d_1}$$
$$=\frac{b_1A_1+b_2A_2+b_3A_3}{-|a_1,\ b_2,\ d_3|}=\frac{c_1A_1+c_2A_2+c_3A_3}{-|a_1,\ c_2,\ d_3|}.$$

Therefore, since $\Sigma b_1A_1\equiv 0$, $\Sigma c_1A_1\equiv 0$, and $A_1\neq 0$,

$$|a_1,\ b_2,\ d_3|=0\ \text{ and }\ |a_1,\ c_2,\ d_3|=0.$$

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The conditions for a line of intersection are often written in the form,

$$\left\| \begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix} \right\| = 0,$$

the notation signifying that any two of the four third-order determinants are zero. They may also be obtained as follows. Any plane through the line of intersection of $u_1=0$, $u_2=0$ is given by $\lambda_1 u_1+\lambda_2 u_2=0$. If the planes $u_1=0$, $u_2=0$, $u_3=0$ pass through one line,

$$\lambda_1 u_1+\lambda_2 u_2=0 \quad \text{and} \quad u_3=0$$

must, for some values of λ_1, λ_2, represent the same plane, and therefore

$$\lambda_1 u_1+\lambda_2 u_2 \equiv -\lambda_3 u_3,$$

or

$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0.$$

Conversely, if λ_1, λ_2, λ_3 can be found so that

$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0,$$

then

$$\lambda_1 u_1+\lambda_2 u_2 \equiv -\lambda_3 u_3,$$

and therefore the plane $u_3=0$ passes through the line of intersection of $u_1=0$ and $u_2=0$. Considering the coefficients in $\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0$, we have

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3=0, \qquad b_1\lambda_1+b_2\lambda_2+b_3\lambda_3=0,$$
$$c_1\lambda_1+c_2\lambda_2+c_3\lambda_3=0, \quad \text{and} \quad d_1\lambda_1+d_2\lambda_2+d_3\lambda_3=0.$$

Therefore, eliminating λ_1, λ_2, λ_3, we obtain

$$\left\| \begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix} \right\| = 0.$$

Ex. 3. Prove that the planes

$$x+ay+(b+c)z+d=0,$$
$$x+by+(c+a)z+d=0,$$
$$x+cy+(a+b)z+d=0$$

pass through one line.

Ex. 4. Prove that the planes $x=cy+bz$, $y=az+cx$, $z=bx+ay$ pass through one line if $a^2+b^2+c^2+2abc=1$.

Ex. 5. The planes $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$ pass through one line if $\Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} =0$, and the direction-ratios of the line satisfy the equations

$$\frac{l^2}{\dfrac{\partial\Delta}{\partial a}}=\frac{m^2}{\dfrac{\partial\Delta}{\partial b}}=\frac{n^2}{\dfrac{\partial\Delta}{\partial c}}.$$

Ex. 6. If the axes are rectangular, the equations to the planes through the line of intersection of two of the given planes

$$a_rx+b_ry+c_rz+d_r=0, \quad r=1,\ 2,\ 3,$$

perpendicular to the third, are

$$(a_1x+b_1y+c_1z+d_1)(a_2a_3+b_2b_3+c_2c_3)-(a_2x+b_2y+c_2z+d_2)$$
$$\times(a_3a_1+b_3b_1+c_3c_1)=0, \text{ etc.}$$

Shew that the three planes pass through one line.

Ex. 7. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes **OX, OY, OZ**, which are rectangular, in **A, B, C.** Prove that the planes through the axes and the internal bisectors of the angles of the triangle **ABC** pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}}=\frac{y}{b\sqrt{c^2+a^2}}=\frac{z}{c\sqrt{a^2+b^2}}.$$

46. Line intersecting two given lines. The equations to any line intersecting two given lines, $u_1=0=v_1$; $u_2=0=v_2$, are

$$u_1+\lambda_1v_1=0, \quad u_2+\lambda_2v_2=0.$$

For the third line lies in the plane $u_1+\lambda_1v_1=0$, and therefore it is coplanar with $u_1=0=v_1$, and similarly it is coplanar with $u_2=0=v_2$.

Ex. 1. Find the equations to the straight line drawn from the origin to intersect the lines

$$3x+2y+4z-5=0=2x-3y+4z+1,$$
$$2x-4y+z+6=0=3x-4y+z-3.$$

Ans. $\frac{x}{249}=\frac{y}{153}=\frac{z}{-52}$.

Ex. 2. Find the equations to the line that intersects the lines $x+y+z=1$, $2x-y-z=2$; $x-y-z=3$, $2x+4y-z=4$, and passes through the point (1, 1, 1). *Ans.* $\frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{3}$.

Ex. 3. Find the equations to the line drawn parallel to $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$ so as to meet the lines $z=5x-6=4y+3$, $z=2x-4=3y+5$.

Ans. $44z=11x+1693$, $11z=11y+345$.

Ex. 4. Find the surface generated by a line which intersects the lines $y=z=a$; $x+3z=a$, $y+z=a$, and is parallel to the plane $x+y=0$.
Ans. $(x+y)(y+z)=2a(z+x)$.

Ex. 5. Find the surface generated by a straight line which intersects the lines $x+y=z=0$; $x-y=z$, $x+y=2a$, and the parabola $y=0$, $x^2=2az$. *Ans.* $x^2-y^2=2az$.

Ex. 6. A variable line intersects OX, and the curve $x=y$, $y^2=cz$, and is parallel to the plane YOZ. Prove that it generates the paraboloid $xy=cz$.

Ex. 7. Prove that the locus of a variable line which intersects the three given lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; $y=z$, $mx=-c$; is the surface $y^2-m^2x^2=z^2-c^2$.

47. Lines intersecting three given lines. If the equations to three given lines are $u_1=0=v_1$, $u_2=0=v_2$, $u_3=0=v_3$, and the three planes

$$(1)\ u_1-\lambda_1 v_1=0,\quad (2)\ u_2-\lambda_2 v_2=0,\quad (3)\ u_3-\lambda_3 v_3=0$$

have a line of intersection, that line is coplanar with each of the three given lines, and therefore intersects all three. There are two independent conditions for a line of intersection, (§ 45), which may be written,

$$f_1(\lambda_1,\ \lambda_2,\ \lambda_3)=0,\ \ldots\ldots\ldots(4)\qquad f_2(\lambda_1,\ \lambda_2,\ \lambda_3)=0.\ \ldots\ldots\ldots(5)$$

If λ_1, λ_2, λ_3 be chosen to satisfy (4) and (5), any two of the equations (1), (2), (3) represent a line which intersects the three given lines. Suppose that (1) and (2) are taken, then eliminating λ_3 between (4) and (5), we obtain

$$\phi(\lambda_1,\ \lambda_2)=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

An infinite number of values of λ_1, λ_2 can be found to satisfy (6), and therefore an infinite number of lines can be found to intersect three given lines. If we eliminate λ_1, λ_2 between (1), (2), (6) we obtain

$$\phi\left(\frac{u_1}{v_1},\ \frac{u_2}{v_2}\right)=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This equation is satisfied by the coordinates of any point on any line which intersects the three given lines, and therefore represents a surface generated by such lines. Hence the lines which intersect three given lines lie on a surface.

It is to be noted that if λ_1, λ_2, λ_3 satisfy (4) and (5), (3) is of the form $u_1-\lambda_1 v_1+k(u_2-\lambda_2 v_2)=0$, and therefore that (1), (2), (3), (4), (5) are really equivalent to four independent equations. The equation to the surface is obtained by eliminating λ_1, λ_2, λ_3 between these four equations, and this can be done in only *one* way. Hence the surface is also given by $f_1\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0$, or by $f_2\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0$.

Ex. 1. Find the locus of lines which intersect the three lines $y=b$, $z=-c$; $z=c$; $x=-a$; $x=a$, $y=-b$.

If the three planes

$$y-b+\lambda_1(z+c)=0, \quad z-c+\lambda_2(x+a)=0, \quad x-a+\lambda_3(y+b)=0$$

have a line of intersection, it meets the three given lines. That is so if

$$\left\|\begin{matrix} 0, & 1, & \lambda_1, & -b+\lambda_1 c \\ \lambda_2, & 0, & 1, & -c+\lambda_2 a \\ 1, & \lambda_3, & 0, & -a+\lambda_3 b \end{matrix}\right\|=0,$$

i.e. if (1) $\lambda_1\lambda_2\lambda_3+1=0$ and (2) $\lambda_1\lambda_2\lambda_3 a-2c\lambda_1\lambda_3+2b\lambda_3-a=0$.

Therefore the coordinates of any point on a line which meets the three given lines satisfy

$$y-b+\lambda_1(z+c)=0, \quad z-c+\lambda_2(x+a)=0, \quad x-a+\lambda_3(y+b)=0,$$

where $\lambda_1\lambda_2\lambda_3+1=0$. Therefore eliminating λ_1, λ_2, λ_3, we obtain the locus of the lines, viz. :

$$\frac{y-b}{z+c}\cdot\frac{z-c}{x+a}\cdot\frac{x-a}{y+b}=1,$$

$$\text{or} \quad ayz+bzx+cxy+abc=0.$$

(Shew that the same result is obtained from (2).)

Ex. 2. If the planes through a point P and the three given lines $y=1$, $z=-1$; $z=1$, $x=-1$; $x=1$, $y=-1$ pass through one line, P lies on the surface $yz+zx+xy+1=0$.

Ex. 3. Prove that all lines which intersect the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; and the x axis, lie on the surface $mxz=cy$.

Ex. 4. Prove that the locus of lines which intersect the three lines $y-z=1$, $x=0$; $z-x=1$, $y=0$; $x-y=1$, $z=0$ is

$$x^2+y^2+z^2-2yz-2zx-2xy=1.$$

Ex. 5. Find the locus of the straight lines which meet the lines

$$x=2,\ 4y=3z\ ; \quad x+2=0,\ 4y+3z=0\ ; \quad y=3,\ 2x+z=0.$$

Ans. $36x^2+16y^2-9z^2=144$.

Ex. 6. Shew that the equations to any line which intersects the three given lines $y=b,\ z=-c,\ z=c,\ x=-a\ ;\ x=a,\ y=-b$ may be written $y-b+\lambda(z+c)=0,\ (x-a)+\mu(y+b)=0$, where λ and μ are connected by the equation $\lambda\mu c-\mu b+a=0$. Hence shew that the two lines which intersect the three given lines and also $\frac{x}{c}=\frac{y+c}{c}=\frac{z-b}{-(a+b)}$ are

$$\frac{x}{a}=\frac{y+c}{c-b}=\frac{z-b}{b-c},\quad \frac{x-c}{c-a}=\frac{y}{b}=\frac{z+a}{a-c}.$$

Ex. 7. Shew that the two lines that can be drawn to intersect the four given lines

$$y=1,\ z=-1\ ;\quad z=1,\ x=-1\ ;\quad x=1,\ y=-1\ ;\quad x=0,\ y+z=0$$

are given by $z=1,\ y+1=0\ ;\ \ z+2x+1=0,\ y-z-2=0.$

48. Coplanar lines. *To find the condition that two given lines should be coplanar.*

Let their equations be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \text{.....................(1)}$$

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}. \quad \text{.....................(2)}$$

The equation to a plane through the first line is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0, \quad \text{..............(3)}$$

where

$$al+bm+cn=0. \quad \text{.......................(4)}$$

If it contains the line (2),

$$a(\alpha-\alpha')+b(\beta-\beta')+c(\gamma-\gamma')=0, \quad \text{............(5)}$$

and

$$al'+bm'+cn'=0. \quad \text{.......................(6)}$$

Therefore eliminating a, b, c between (4), (5), (6), we obtain the required condition,

$$\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0. \quad \text{.............(7)}$$

The elimination of a, b, c between (3), (4), (6) gives the equation to the plane containing the lines, viz.,

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0. \quad \text{..............(8)}$$

Generally, the equation (8) represents the plane through the line (1) parallel to the line (2), and (7) is the condition that this plane should contain the point $(\alpha', \beta', \gamma')$ on (2).

Ex. 1. Deduce the result (7) by equating the coordinates $\alpha + lr$, etc., $\alpha' + l'r'$, etc., of variable points on the given lines.

Ex. 2. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Ex. 3. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta};$$

$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$

are coplanar, and find the equation to the plane in which they lie.
Ans. $2y = x + z$.

Ex. 4. Prove that the lines $x = ay + b = cz + d$, $x = \alpha y + \beta = \gamma z + \delta$ are coplanar if $(\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0$.

Ex. 5. Prove that the lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \quad ax + by + cz + d = 0 = a'x + b'y + c'z + d'$$

are coplanar if $\dfrac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \dfrac{a'\alpha + b'\beta + c'\gamma + d'}{a'l + b'm + c'n}$.

Ex. 6. Prove that the lines $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$; $\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$ are coplanar if

$$\begin{vmatrix} a, & a', & \alpha, & \alpha' \\ b, & b', & \beta, & \beta' \\ c, & c', & \gamma, & \gamma' \\ d, & d', & \delta, & \delta' \end{vmatrix} = 0.$$

Ex. 7. A, A′; B, B′; C, C′ are points on the axes; shew that the lines of intersection of the planes A′BC, AB′C′; B′CA, BC′A′; C′AB, CA′B′ are coplanar.

49. The shortest distance between two lines. *The axes being rectangular, to find the shortest distance between the lines* $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$; $\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$.

Let the points A, A′, (fig. 24), be (α, β, γ), $(\alpha', \beta', \gamma')$. The shortest distance between the given lines is at right angles to both, and it is therefore equal to the projection of AA′ on a line at right angles to both of the given lines.

Suppose that λ, μ, ν are the direction-cosines of such a line, then $l\lambda+m\mu+n\nu=0$ and $l'\lambda+m'\mu+n'\nu=0$;

$$\therefore \frac{\lambda}{mn'-m'n}=\frac{\mu}{nl'-n'l}=\frac{\nu}{lm'-l'm}.$$

Therefore the projection

$$=\lambda(\alpha-\alpha')+\mu(\beta-\beta')+\nu(\gamma-\gamma'),\quad (\S\, 21, \text{ Ex. } 3),$$

$$=\frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(nl'-n'l)+(\gamma-\gamma')(lm'-l'm)}{\sqrt{\Sigma(mn'-m'n)^2}},$$

$$=\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \div \sqrt{\Sigma(mn'-m'n)^2}.$$

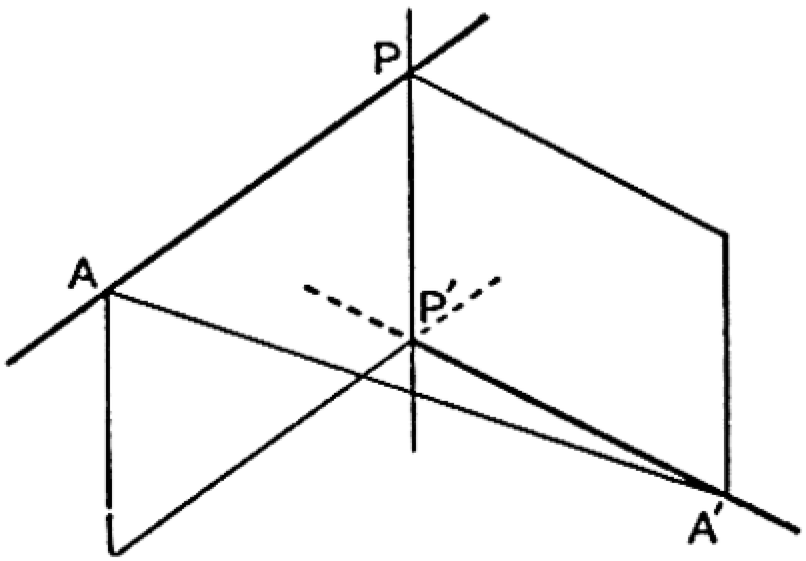

FIG. 24.

Equation (7), § 48, may now be interpreted as the condition that the shortest distance between the two given lines should vanish. Again, if PP' is the shortest distance, the equations to the planes APP', A'PP' are

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ \lambda, & \mu, & \nu \end{vmatrix}=0, \quad \begin{vmatrix} x-\alpha', & y-\beta', & z-\gamma' \\ l', & m', & n' \\ \lambda, & \mu, & \nu \end{vmatrix}=0,$$

and these represent the line PP'.

Ex. 1. Find the shortest distance using the theorems that the shortest distance is equal to (i), the perpendicular from any point $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$, on the first line to the plane drawn through the second parallel to the first; and (ii), the distance between two planes, each passing through one line and parallel to the other.

Ex. 2. If P, $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$ and

P', $(\alpha'+l'r',\ \beta'+m'r',\ \gamma'+n'r')$

are points on the given lines, and PP'$=\delta$, prove that $\frac{\partial\delta^2}{\partial r}=0$, $\frac{\partial\delta^2}{\partial r'}=0$, necessary conditions for a minimum of PP'^{2}, are verified when PP' is perpendicular to each of the lines.

Ex. 3. Shew that the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}\,;\quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

is $\frac{1}{\sqrt{6}}$, and that its equations are

$$11x+2y-7z+6=0,\quad 7x+y-5z+7=0.$$

Ex. 4. Find the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}\,;\quad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

The following method of solution may be adopted: Let the S.D. meet the lines in P and P′ respectively. Then the coordinates of P and P′ may be written $(3+3r,\ 8-r,\ 3+r)$, $(-3-3r',\ -7+2r',\ 6+4r')$, where r is proportional to the distance of P from the point (3, 8, 3) and r' to the distance of P′ from (−3, −7, 6). Whence the direction-cosines of PP′ are proportional to $6+3r+3r'$, $15-r-2r'$, $-3+r-4r'$.

Since PP′ is at right angles to both lines, we have

$$3(6+3r+3r')-(15-r-2r')+(-3+r-4r')=0,$$

$$-3(6+3r+3r')+2(15-r-2r')+4(-3+r-4r')=0.$$

Whence, solving for r and r', we get $r=r'=0$.

Therefore P and P′ are the points (3, 8, 3), (−3, −7, 6), $PP'=3\sqrt{30}$, and the equations to PP′ are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}.$$

Ex. 5. Find the same results for the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}\,;\quad \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$$

Ans. $2\sqrt{29}$, $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, (3, 5, 7), (−1, −1, −1).

Ex. 6. Find the length and equations of the S.D. between

$$3x-9y+5z=0=x+y-z,$$

$$6x+8y+3z-13=0=x+2y+z-3.$$

Ans. $\frac{11}{\sqrt{342}}$, $10x-29y+16z=0=13x+82y+55z-109$.

Ex. 7. A line with direction-cosines proportional to 2, 7, −5 is drawn to intersect the lines

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}\,;\quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the points of intersection and the length intercepted on it.

Ans. (2, 8, −3), (0, 1, 2), $\sqrt{78}$.

Ex. 8. Find the S.D. between the axis of z and the line

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0.$$

(The plane passing through the line and parallel to **OZ** is

$$c'(ax+by+cz+d)=c(a'x+b'y+c'z+d'),$$

and the perpendicular from the origin to this plane is equal to the S.D.)

Ans. $\dfrac{cd'-c'd}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}$.

Ex. 9. If the axes are rectangular, the S.D. between the lines $y=az+b$, $z=\alpha x+\beta$; $y=a'z+b'$, $z=\alpha' x+\beta'$ is

$$\frac{(\alpha-\alpha')(b-b')-(\alpha'\beta-\alpha\beta')(a-a')}{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a\alpha-a'\alpha')^2\}^{\frac{1}{2}}}.$$

Ex. 10. Prove that the S.D. between the lines

$$ax+by+cz+d=0=a'x+b'y+c'z+d',$$
$$\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$$

is
$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{vmatrix} \div \{\Sigma(\mathsf{BC}'-\mathsf{B}'\mathsf{C})^2\}^{\frac{1}{2}},$$

where $\mathsf{A} \equiv bc'-b'c$, etc , $\mathsf{A}' \equiv \beta\gamma'-\beta'\gamma$, etc.

Ex. 11. Shew that the S.D. between the lines

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1}; \quad \frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_3}{\cos\gamma_2}$$

meets the first line at a point whose distance from (x_1, y_1, z_1) is $\dfrac{\Sigma(x_1-x_2)(\cos\alpha_1-\cos\theta\cos\alpha_2)}{\sin^2\theta}$, where θ is the angle between the lines.

Ex. 12. Shew that the S.D. between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $2a/\sqrt{6}$, and that the three lines of shortest distance intersect at the point $x=y=z=-a$.

Ex. 13. Shew that the S.D. between the line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$

and the z-axis meets the z-axis at a point whose distance from the origin is

$$\frac{db'-d'b)(bc'-b'c)+(ca'-c'a)(ad'-a'd)}{\{(bc'-b'c)^2+(ca'-c'a)^2\}}.$$

Ex. 14. Shew that the equation to the plane containing the line $y/b+z/c=1$, $x=0$; and parallel to the line $x/a-z/c=1$, $y=0$ is $x/a-y/b-z/c+1=0$, and if $2d$ is the S.D. prove that $\dfrac{1}{d^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$.

Ex. 15. Two straight lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}; \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

are cut by a third whose direction-cosines are λ, μ, ν. Shew that the length intercepted on the third line is given by

$$\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \div \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \lambda, & \mu, & \nu \end{vmatrix},$$

and deduce the length of the S.D.

***Ex. 16.** The axes are oblique and the plane **ABC** has equation $x/a+y/b+z/c=1$. Prove that if the tetrahedron **OABC** has two pairs of opposite edges at right angles, $\frac{\cos\lambda}{a}=\frac{\cos\mu}{b}=\frac{\cos\nu}{c}$ $(=k)$, and that the equations to the four perpendiculars are

$$\frac{\partial\phi}{\partial y}=2a\cos\nu, \ \frac{\partial\phi}{\partial z}=2a\cos\mu, \text{ etc., and } a\frac{\partial\phi}{\partial x}=b\frac{\partial\phi}{\partial y}=c\frac{\partial\phi}{\partial z}.$$

Hence shew that the perpendiculars pass through the point given by $\frac{\partial\phi}{\partial x}=2bck$, $\frac{\partial\phi}{\partial y}=2cak$, $\frac{\partial\phi}{\partial z}=2abk$. Prove also that the equations to the S.D. of **AB** and **OC** are $\frac{\partial\phi}{\partial z}=2abk$, $a\frac{\partial\phi}{\partial x}=b\frac{\partial\phi}{\partial y}$; and that the S.D passes through the point of concurrence of the perpendiculars.

50. Problems relating to two non-intersecting lines.

When two non-intersecting lines are given, the following systems of coordinate axes allow their equations to be written in simple forms, and are therefore of use in problems relating to the lines.

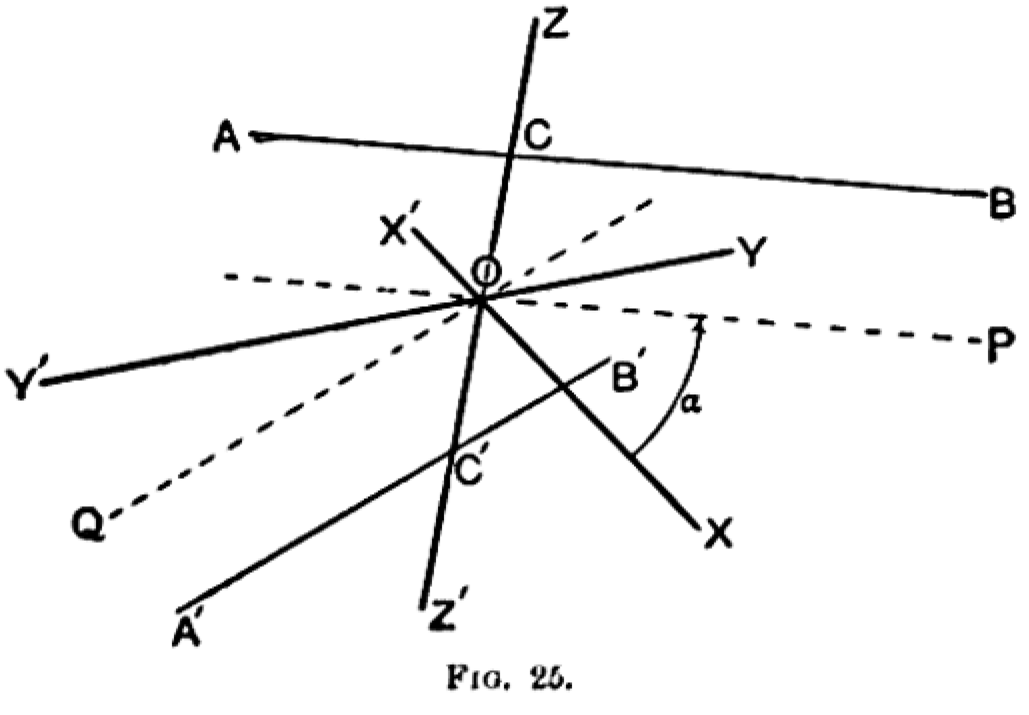

FIG. 25.

I. *Rectangular axes.* Let **AB**, **A'B'**, (fig. 25), be the lines, and let **CC'**, length $2c$, be the shortest distance between them. Take the axis of z along **CC'**, and **O** the mid-point of CC′ as origin. Draw OP, OQ parallel to AB, A′B′, and take the plane POQ as the plane $z=0$. As x- and y-axes take the bisectors of the angles between OP and OQ. Then if the angle between the given lines is 2α, the equations to the planes POZ, QOZ are $y=x\tan\alpha$, $y=-x\tan\alpha$; and hence the equations to AB and A′B′ are

$$y=x\tan\alpha,\ z=c;\quad y=-x\tan\alpha,\ z=-c.$$

These may be written in the symmetrical forms

$$\frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=\frac{z-c}{0};\quad \frac{x}{\cos\alpha}=\frac{y}{-\sin\alpha}=\frac{z+c}{0}.$$

Ex. 1. P and P′ are variable points on two given non-intersecting lines AB and A′B′, and Q is a variable point so that QP, QP′ are at right angles to one another and at right angles to AB and A′B′ respectively. Find the locus of Q.

Take as the equations to AB, A′B′, $y=mx$, $z=c$; $y=-mx$, $z=-c$. Then the coordinates of P, P′ are α, $m\alpha$, c; β, $-m\beta$, $-c$, where α and β are variables. Let Q be (ξ, η, ζ), then since PQ is perpendicular to AB,

$$(\xi-\alpha)+m(\eta-m\alpha)=0;\ \dots\dots(1)$$

since P′Q is perpendicular to A′B′,

$$(\xi-\beta)-m(\eta+m\beta)=0;\ \dots\dots(2)$$

since PQ is perpendicular to P′Q,

$$(\xi-\alpha)(\xi-\beta)+(\eta-m\alpha)(\eta+m\beta)+(\zeta-c)(\zeta+c)=0.\ \dots\dots(3)$$

To find the equation to the locus we have to eliminate α and β between (1), (2), (3).

The result is easily found to be $\dfrac{m^2\xi^2-\eta^2}{(1+m^2)^2}=\dfrac{\zeta^2-c^2}{1-m^2}$, which represents a hyperboloid.

II. *Axes partly rectangular.* If we take OP and OQ as axes of x and y, instead of the bisectors of the angles between them, we have a system of axes in which the angles ZOX, YOZ are right angles and the angle XOY is the angle between the lines. The equations to AB, A′B′ referred to this system are

$$y=0,\ z=c;\quad x=0,\ z=-c.$$

Ex. 2. P, P′ are variable points on two given non-intersecting lines and PP′ is of constant length $2k$. Find the surface generated by PP′.

Take as the equations to the lines $y=0$, $z=c$; $x=0$, $z=-c$; then P and P′ are $(\alpha, 0, c)$, $(0, \beta, -c)$, where α and β are variables. The equations to PP′ are

$$\frac{x}{\alpha}=\frac{y-\beta}{-\beta}=\frac{z+c}{2c}.\ \dots\dots(1)$$

If Q, Q′ are the projections of P, P′ on the plane OXY, $PQ=Q'P'=c$, $OQ=\alpha$, $OQ'=\beta$ and $QQ'^2=\alpha^2+\beta^2-2\alpha\beta\cos\theta$, where θ is the angle between the lines. Therefore

$$PP'^2=\alpha^2+\beta^2-2\alpha\beta\cos\theta+4c^2=4k^2. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

To obtain the equation to the locus of PP′ we have to eliminate α and β between the equations (1) and (2). From (1),

$$\alpha=\frac{2cx}{z+c},\quad \beta=\frac{-2cy}{z-c},$$

and therefore the surface is given by

$$\frac{x^2}{(z+c)^2}+\frac{y^2}{(z-c)^2}+\frac{2xy\cos\theta}{z^2-c^2}=\frac{k^2}{c^2}-1.$$

Ex. 3. Find the surface generated by a straight line which intersects two given lines and is parallel to a given plane.

If the axes be chosen as in Ex. 2, and the given plane be $lx+my+nz=0$, the locus is $\dfrac{lx}{z+c}+\dfrac{my}{z-c}+n=0$.

III. *Axes oblique.* If a point on each of the given lines is specified and a rectangular system is not necessary, the line joining the given points may be taken as z-axis, its mid-point as origin, and the parallels through the origin to the given lines as x- and y-axes. The equations to the lines are then

$$y=0,\ z=c;\quad x=0,\ z=-c;$$

where $2c$ is the distance between the given points.

Ex. 4. AP, A′P′ are two given lines, A and A′ being fixed, and P and P′ variable points such that AP . A′P′ is constant. Find the locus of PP′.

Take AA′ as z-axis, etc. Then P, P′ are $(\alpha, 0, c)$, $(0, \beta, -c)$, where $\alpha\beta=\text{constant}=4k^2$, say. The equations to PP′ are

$$\frac{x}{\alpha}=\frac{y-\beta}{-\beta}=\frac{z+c}{2c},$$

and eliminating α and β between these and $\alpha\beta=4k^2$, we obtain the equation to the locus, $c^2xy+k^2(z^2-c^2)=0$.

Ex. 5. Find the locus of PP′ when (i) AP+A′P′, (ii) AP/A′P′, (iii) $AP^2+A'P'^2$ is constant. Find also the locus of the mid-point of PP′.

Ex. 6. Find the locus of the mid-points of lines whose extremities are on two given lines and which are parallel to a given plane.

Ex. 7. Find the locus of a straight line that intersects two given lines and makes a right angle with one of them.

Ex. 8. Find the locus of a point which is equidistant from two given straight lines.

Ex. 9. Shew that the locus of the mid-points of lines of constant length which have their extremities on two given lines is an ellipse whose centre bisects the S.D., and whose axes are equally inclined to the lines.

Ex. 10. A point moves so that the line joining the feet of the perpendiculars from it to two given lines subtends a right angle at the mid-point of their S.D. Shew that its locus is a hyperbolic cylinder.

Ex. 11. Prove that the locus of a line which meets the lines $y = \pm mx$, $z = \pm c$; and the circle $x^2 + y^2 = a^2$, $z = 0$ is

$$c^2m^2(cy - mxz)^2 + c^2(yz - cmx)^2 = a^2m^2(z^2 - c^2)^2.$$

THE VOLUME OF A TETRAHEDRON.

51. *To find the volume in terms of the coordinates of the vertices, the axes being rectangular.*

If A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), the equation to the plane ABC is $\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0$, or

$$x\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix} + y\begin{vmatrix} z_1, & x_1, & 1 \\ z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \end{vmatrix} + z\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \quad \text{....(1)}$$

The equation to the plane ABC can also be written

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma. \quad \text{..............(2)}$$

Let Δ denote the area ABC; then its projections on the planes YOZ, ZOX, XOY are $\cos\alpha.\Delta$, $\cos\beta.\Delta$, $\cos\gamma.\Delta$ respectively. But the projections of A, B, C on the plane YOZ are $(0, y_1, z_1)$, $(0, y_2, z_2)$, $(0, y_3, z_3)$, and therefore the area of the projection of ABC is given in magnitude and sign by $\frac{1}{2}\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}$. Therefore we have

$$\cos\alpha.\Delta = \tfrac{1}{2}\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}, \text{ in magnitude and sign.}$$

Hence, using the similar expressions for $\cos\beta \,.\, \Delta$ and $\cos\gamma \,.\, \Delta$, equation (1) may be written

$$2\Delta(x\cos\alpha+y\cos\beta+z\cos\gamma)=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}=2p\Delta, \text{ by (2).}$$

Now the absolute measure of $\frac{1}{3}p\Delta$ is the volume of the tetrahedron OABC, and we can introduce positive and negative volume by defining the volume OABC to be $\frac{1}{3}p\Delta$, which is positive or negative according as the direction of rotation determined by ABC is positive or negative for the plane ABC, (p is positive as in § 37). We may then write

$$\text{Vol. OABC}=\text{Vol. OCAB}=\text{Vol. OBCA}=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix};$$

$$\text{Vol. OBAC}=\tfrac{1}{6}\begin{vmatrix} x_2, & y_2, & z_2 \\ x_1, & y_1, & z_1 \\ x_3, & y_3, & z_3 \end{vmatrix}=-\text{Vol. OABC, etc.}$$

If D is the point (x_4, y_4, z_4), changing the origin to D, we have

$$\text{Vol. DABC}=\tfrac{1}{6}\begin{vmatrix} x_1-x_4, & y_1-y_4, & z_1-z_4 \\ x_2-x_4, & y_2-y_4, & z_2-z_4 \\ x_3-x_4, & y_3-y_4, & z_3-z_4 \end{vmatrix}$$

$$=\tfrac{1}{6}\begin{vmatrix} x_1-x_4, & y_1-y_4, & z_1-z_4, & 0 \\ x_2-x_4, & y_2-y_4, & z_2-z_4, & 0 \\ x_3-x_4, & y_3-y_4, & z_3-z_4, & 0 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

$$=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}=-\tfrac{1}{6}\begin{vmatrix} x_4, & y_4, & z_4, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}.$$

Since the sign of a determinant is changed when two adjacent rows or columns are interchanged, it follows that

$$\text{Vol. DABC}=-\text{Vol. ADBC}=\text{Vol. ABDC}=-\text{Vol. ABCD, etc}$$

Again, since

$$\begin{vmatrix} x_1, y_1, z_1, 1 \\ x_2, y_2, z_2, 1 \\ x_3, y_3, z_3, 1 \\ x_4, y_4, z_4, 1 \end{vmatrix} \equiv \begin{vmatrix} x_1, y_1, z_1 \\ x_2, y_2, z_2 \\ x_3, y_3, z_3 \end{vmatrix} - \begin{vmatrix} x_2, y_2, z_2 \\ x_3, y_3, z_3 \\ x_4, y_4, z_4 \end{vmatrix} + \begin{vmatrix} x_3, y_3, z_3 \\ x_4, y_4, z_4 \\ x_1, y_1, z_1 \end{vmatrix} - \begin{vmatrix} x_4, y_4, z_4 \\ x_1, y_1, z_1 \\ x_2, y_2, z_2 \end{vmatrix};$$

$$-\text{Vol. }\mathsf{ABCD} = \text{Vol. }\mathsf{OABC} - \text{Vol. }\mathsf{OBCD} + \text{Vol. }\mathsf{OCDA} - \text{Vol. }\mathsf{ODAB}.$$

Since the volume **ABCD** does not depend on the position of the origin, this must be true for all positions of **O**.

Cor. If α, β, γ, δ are the perpendiculars from any point **O** to the faces **ABC**, **BCD**, **CDA**, **DAB** of a given tetrahedron $a\alpha+b\beta+c\gamma+d\delta$ is constant, where a, b, c, d are certain constants.

Ex. 1. **A**, **B**, **C** are (3, 2, 1), $(-2, 0, -3)$, $(0, 0, -2)$. Find the locus of **P** if the vol. **PABC** $=5$. *Ans.* $2x+3y-4z=38$.

Ex. 2. The lengths of two opposite edges of a tetrahedron are a, b, their S.D. is equal to d, and the angle between them to θ; prove that the volume is $\dfrac{abd\sin\theta}{6}$.

Ex. 3. **AA′** is the S.D. between two given lines, and **B**, **B′** are variable points on them such that the volume **AA′BB′** is constant. Prove that the locus of the mid-point of **BB′** is a hyperbola whose asymptotes are parallel to the lines.

Ex. 4. If **O**, **A**, **B**, **C**, **D** are any five points, and p_1, p_2, p_3, p_4 are the projections of **OA**, **OB**, **OC**, **OD** on any given line, prove that

$$p_1 \,.\, \text{Vol. }\mathsf{OBCD} - p_2 \,.\, \text{Vol. }\mathsf{OCDA} + p_3 \,.\, \text{Vol. }\mathsf{ODAB} - p_4 \,.\, \text{Vol. }\mathsf{OABC} = 0.$$

Ex. 5. Prove that the volume of a tetrahedron, two of whose sides are of constant length and lie upon given straight lines, is constant, and that the locus of its centre of gravity is a plane.

Ex. 6. If **A**, **B**, **C**, **D** are coplanar and **A′**, **B′**, **C′**, **D′** are their projections on any plane, prove that Vol. **AB′C′D′** $=-$ Vol. **A′BCD**.

Ex. 7. Lines are drawn in a given direction through the vertices **A**, **B**, **C**, **D** of a tetrahedron to meet the opposite faces in **A′**, **B′**, **C′**, **D′**. Prove that Vol. **A′B′C′D′** $=-3$. Vol. **ABCD**.

***Ex. 8.** Find the volume of the tetrahedron the equations to whose faces are $a_rx+b_ry+c_rz+d_r=0$, $r=1, 2, 3, 4$.

Let the planes corresponding to $r=1, 2, 3, 4$ be **BCD**, **CDA**, **DAB**, **ABC** respectively, and let $\Delta \equiv \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$. Then (x_1, y_1, z_1), the coordinates of A_1 are given by

$$\frac{x_1}{\mathsf{A}_1}=\frac{y_1}{\mathsf{B}_1}=\frac{z_1}{\mathsf{C}_1}=\frac{1}{\mathsf{D}_1},\ \text{where } \mathsf{A}_1=\frac{\partial\Delta}{\partial a_1},\ \text{etc.};$$

and therefore the volume is given by

$$-\tfrac{1}{6}\begin{vmatrix} \mathsf{A}_1/\mathsf{D}_1, & \mathsf{B}_1/\mathsf{D}_1, & \mathsf{C}_1/\mathsf{D}_1, & 1 \\ \mathsf{A}_2/\mathsf{D}_2, & & \text{etc.} & \end{vmatrix} = -\frac{1}{6\mathsf{D}_1\mathsf{D}_2\mathsf{D}_3\mathsf{D}_4}\begin{vmatrix} \mathsf{A}_1, & \mathsf{B}_1, & \mathsf{C}_1, & \mathsf{D}_1 \\ \mathsf{A}_2, & & \text{etc.} & \end{vmatrix}$$

$$=\frac{-\Delta^3}{6\mathsf{D}_1\mathsf{D}_2\mathsf{D}_3\mathsf{D}_4}.$$

(C. Smith, *Algebra*, p. 544.)

***Ex. 9.** The lengths of the edges OA, OB, OC of a tetrahedron OABC are a, b, c, and the angles BOC, COA, AOB are λ, μ, ν; find the volume.

Suppose that the direction-cosines of OA, OB, OC, referred to rectangular axes through O, are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; then the coordinates of A are l_1a, m_1a, n_1a, etc.

Therefore
$$6\,.\,\text{Vol. OABC}=\begin{vmatrix} l_1a, & m_1a, & n_1a \\ l_2b, & m_2b, & n_2b \\ l_3c, & m_3c, & n_3c \end{vmatrix}$$

$$=\pm abc\begin{vmatrix} \Sigma l_1^2, & \Sigma l_1l_2, & \Sigma l_1l_3 \\ \Sigma l_1l_2, & \Sigma l_2^2, & \Sigma l_2l_3 \\ \Sigma l_1l_3, & \Sigma l_2l_3, & \Sigma l_3^2 \end{vmatrix}^{\frac{1}{2}}=\pm abc\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

(Cf. § 27, Ex. 3.

CHAPTER IV.

CHANGE OF AXES.

52. OX, OY, OZ ; $O\xi$, $O\eta$, $O\zeta$ are two sets of rectangular axes through a common origin O, and the direction-cosines of $O\xi$, $O\eta$, $O\zeta$, referred to OX, OY, OZ, are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. P, any point, has coordinates x, y, z referred to OX, OY, OZ and ξ, η, ζ referred to $O\xi$, $O\eta$, $O\zeta$. We have to express x, y, z in terms of ξ, η, ζ and the direction-cosines, and *vice-versa*.

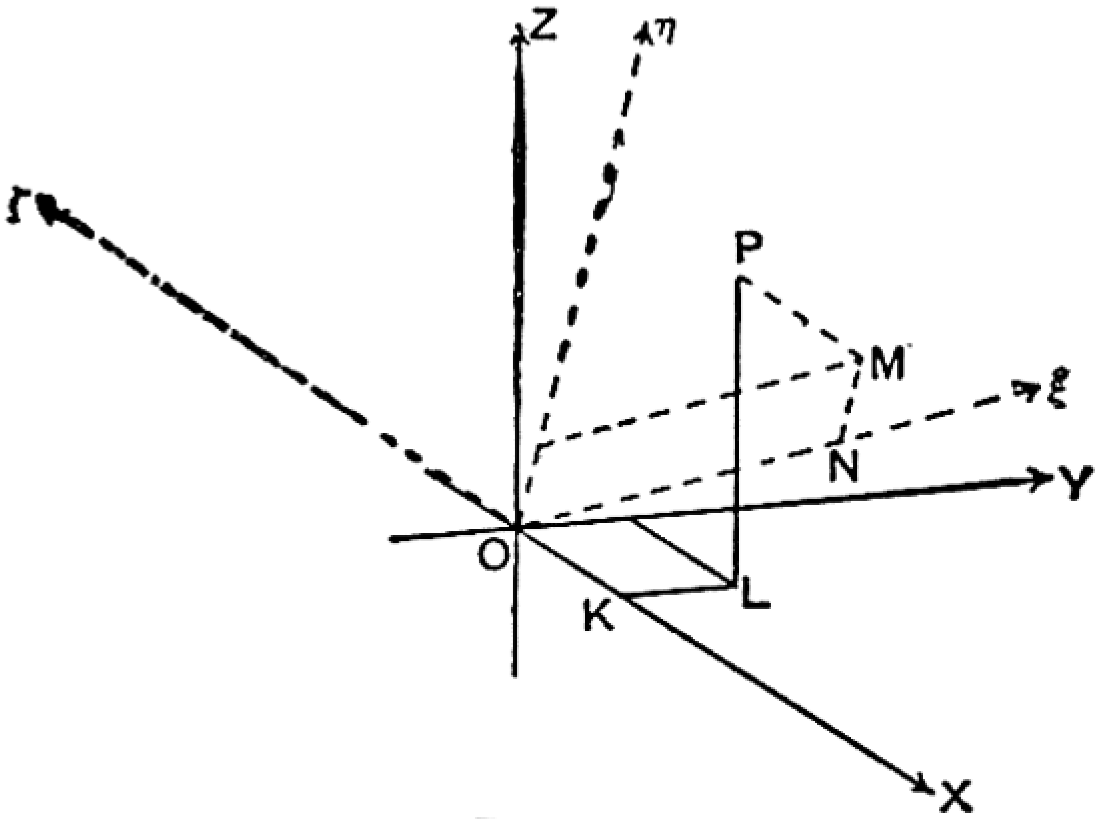

FIG. 26.

In the accompanying figure, ON, NM, MP represent ξ, η, ζ, and OK, KL, LP represent x, y, z. Projecting OP and ON, NM, MP on OX, OY, OZ in turn, we obtain

$$\left.\begin{aligned} x &= l_1\xi + l_2\eta + l_3\zeta, \\ y &= m_1\xi + m_2\eta + m_3\zeta, \\ z &= n_1\xi + n_2\eta + n_3\zeta. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(1)$$

And projecting OP and OK, KL, LP on Oξ, Oη, Oζ in turn, we obtain

$$\left.\begin{aligned} \xi &= l_1x+m_1y+n_1z, \\ \eta &= l_2x+m_2y+n_2z, \\ \zeta &= l_3x+m_3y+n_3z. \end{aligned}\right\} \quad \text{..................(2)}$$

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

The equations (1) and (2) can be derived from the above scheme, which may be constructed as follows: Affix to the columns and rows the numbers x, y, z; ξ, η, ζ; and in the square common to the column headed x and the row headed ξ, place the cosine of the angle between OX and Oξ, *i.e.* l_1, and so on. To obtain the value of x, multiply the numbers in the x-column by the numbers at the left of their respective rows and add the products; to obtain the value of ξ, multiply the numbers in the ξ-row by the numbers at the heads of their respective columns, and add the products. Similarly, any other of the equations (1) and (2) may be derived.

Cor. Since x, y, z are linear functions of ξ, η, ζ, the degree of an equation is unaltered by transformation from any one set of rectangular axes to any other. For it is evident that the degree cannot be raised. Neither can it be lowered, since in changing again to the original axes, it would require to be raised.

53. Relations between the direction-cosines of three mutually perpendicular lines. We have

$$\left.\begin{aligned} l_1^2+m_1^2+n_1^2&=1, \\ l_2^2+m_2^2+n_2^2&=1, \\ l_3^2+m_3^2+n_3^2&=1, \end{aligned}\right\} \text{......(A)} \qquad \left.\begin{aligned} l_2l_3+m_2m_3+n_2n_3&=0, \\ l_3l_1+m_3m_1+n_3n_1&=0, \\ l_1l_2+m_1m_2+n_1n_2&=0. \end{aligned}\right\} \text{......(B)}$$

From the second and third equations of (B), we derive

$$\frac{l_1}{m_2n_3-m_3n_2}=\frac{m_1}{n_2l_3-n_3l_2}=\frac{n_1}{l_2m_3-l_3m_2};$$

$$\text{and each}=\frac{l_1^2+m_1^2+n_1^2}{\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}}=\frac{\pm\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{\Sigma(m_2n_3-m_3n_2)^2}}=\pm 1.$$

(§ 23, *Cor.* 1.)

Therefore, if $\mathsf{D}^{-1}\equiv\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=\pm 1,$

$$\left.\begin{array}{l} l_1=\mathsf{D}(m_2n_3-m_3n_2),\ m_1=\mathsf{D}(n_2l_3-n_3l_2),\ n_1=\mathsf{D}(l_2m_3-l_3m_2). \\ \text{Similarly,} \\ l_2=\mathsf{D}(m_3n_1-m_1n_3),\ m_2=\mathsf{D}(n_3l_1-n_1l_3),\ n_2=\mathsf{D}(l_3m_1-l_1m_3); \\ l_3=\mathsf{D}(m_1n_2-m_2n_1),\ m_3=\mathsf{D}(n_1l_2-n_2l_1),\ n_3=\mathsf{D}(l_1m_2-l_2m_1). \end{array}\right\}\text{(E)}$$

Multiplying the first column of equations (E) by l_1, l_2, l_3 respectively, and adding, we obtain

$$l_1^2+l_2^2+l_3^2=\mathsf{D}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=1;$$

and similarly,

$$m_1^2+m_2^2+m_3^2=1,\quad n_1^2+n_2^2+n_3^2=1.\ \ldots\ldots\ldots\text{(C)}$$

Multiplying the second column by n_1, n_2, n_3, we obtain in the same way,

$$m_1n_1+m_2n_2+m_3n_3=0,$$

and similarly,

$$n_1l_1+n_2l_2+n_3l_3=0,$$

$$l_1m_1+l_2m_2+l_3m_3=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(D)}$$

The equations (C) and (D) can be derived at once from the consideration that l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 are the direction-cosines of OX, OY, OZ referred to $\mathsf{O}\xi$, $\mathsf{O}\eta$, $\mathsf{O}\zeta$. The method adopted shews that the four sets (A), (B), (C), (D) are not independent, and it can be shewn as above, that if either of the two dissimilar sets (A), (B); (C), (D) be given, the other two can be deduced.

Suppose that a plane LMN (fig. 27) cuts off three positive segments of unit length from the axes $\mathsf{O}\xi$, $\mathsf{O}\eta$, $\mathsf{O}\zeta$. Then if the direction of rotation given by LMN is the positive direction of rotation for the plane LMN, the system of axes Oξ, Oη, Oζ can be brought by rotation about O into coincidence with the system OX, OY, OZ. If the direction of rotation is negative, and Oξ, Oη are brought to coincide with OX, OY respectively, then Oζ coincides with OZ′.

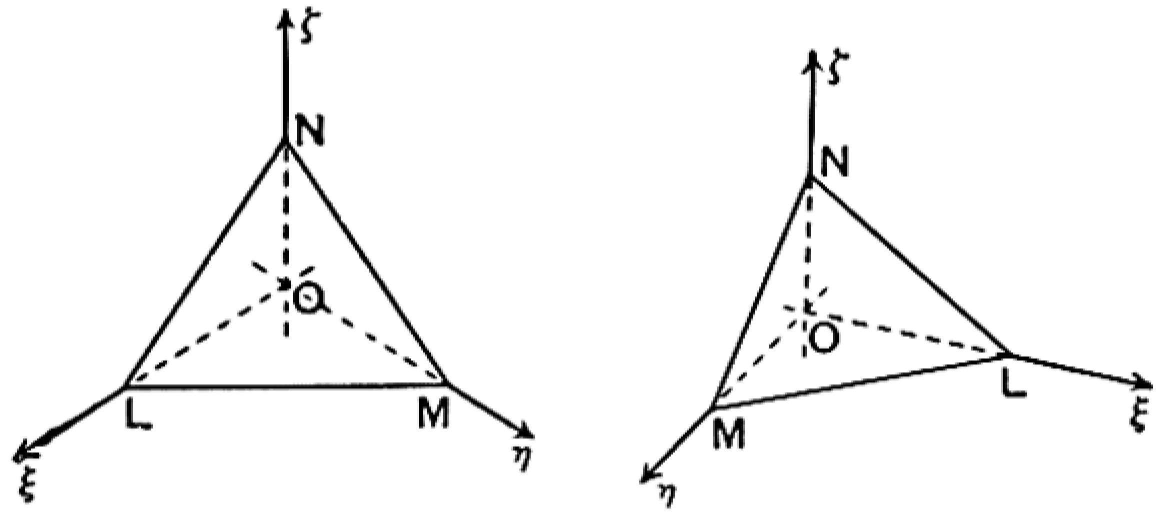

FIG. 27.

Now
$$6\,.\,\text{Vol. OLMN} = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \equiv D^{-1}.$$

The volume, and therefore D, is positive or negative according as the direction of rotation determined by LMN is positive or negative for the plane LMN. Hence if LMN gives the positive direction of rotation, from equations (E)

$$l_1 = +(m_2n_3 - m_3n_2), \text{ etc.,}$$

the positive sign being taken throughout. If LMN gives the negative direction of rotation,

$$l_1 = -(m_2n_3 - m_3n_2), \text{ etc.,}$$

the negative sign being taken throughout.

Conversely, if l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular directed lines Oξ, Oη, Oζ, and

$$l_1 = +(m_2n_3 - m_3n_2),$$

then Oξ, Oη, Oζ can be brought by rotation about O to coincide with OX, OY, OZ.

Ex. Verify the above results by considering Oξ, Oη, Oζ to coincide, say, with OX′, OY′, OZ.

Here $l_1 = -1$, $m_1 = n_1 = 0$; $l_2 = n_2 = 0$, $m_2 = -1$; $l_3 = m_3 = 0$, $n_3 = 1$. $l_1 = m_2n_3 - m_3n_2$, and if Oξ be rotated to coincide with OX, Oη coincides with OY.

54.[1] **Section of a surface by a given plane.** The following method of transformation can be applied with advantage when the section of a given surface by a given plane passing through the origin is to be considered.

Let the equation to the plane be $lx+my+nz=0$, where $l^2+m^2+n^2=1$, and n is positive.

Take as $O\zeta$, the new axis of z, the normal to the plane which passes through O and makes an acute angle with OZ. Then the equations to $O\zeta$, referred to OX, OY, OZ, are $x/l=y/m=z/n$. Take as $O\eta$, the new y-axis, the line in the plane $ZO\zeta$ which is at right angles to $O\zeta$ and makes an acute angle with OZ. Then choose $O\xi$, the new x-axis, at right angles to $O\eta$ and $O\zeta$, and so that the system $O\xi$, $O\eta$, $O\zeta$ can be brought to coincidence with OX, OY, OZ. The given plane is $\xi O\eta$, and since $O\xi$ is at right angles to $O\zeta$ and $O\eta$, it is at right angles to OZ which lies in the plane $\zeta O\eta$. Hence $O\xi$ lies in the plane XOY, and therefore is the line of intersection of the given plane and the plane XOY. The equation to the plane $\zeta O\eta$ is $x/l=y/m$; therefore if λ, μ, ν are the direction-cosines of $O\eta$,

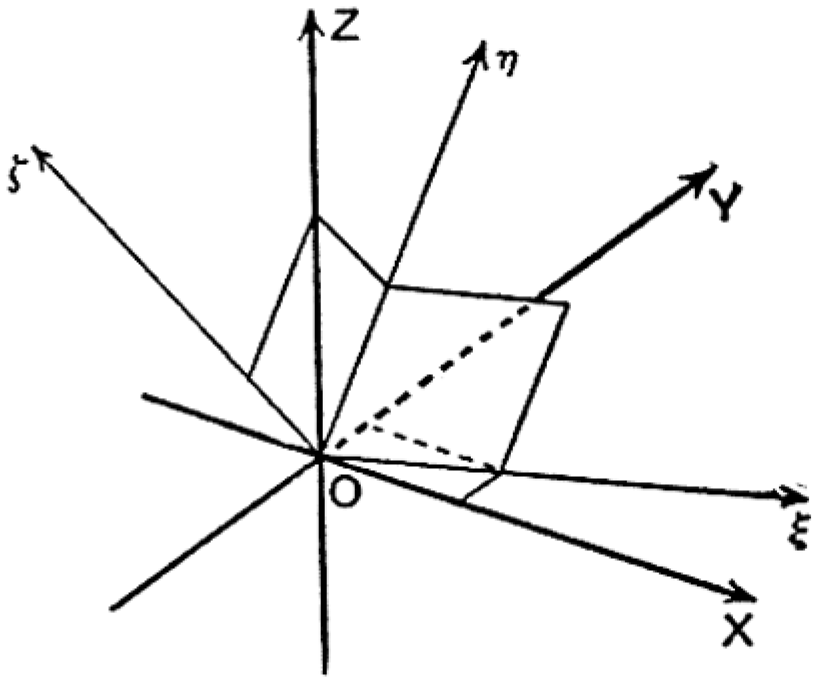

FIG. 28.

$$l\lambda+m\mu+n\nu=0,$$

$$m\lambda-l\mu=0,$$

whence $$\frac{\lambda}{l}=\frac{\mu}{m}=\frac{\nu}{\dfrac{l^2+m^2}{-n}}=\frac{\pm 1}{\dfrac{\sqrt{l^2+m^2}}{n}}.$$

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[1] See Appendix, p. ii.

But Oη makes an acute angle with OZ, and therefore ν is positive, and therefore the negative sign must be taken in the ambiguity.

$$\therefore\ \lambda=\frac{-ln}{\sqrt{l^2+m^2}},\quad \mu=\frac{-mn}{\sqrt{l^2+m^2}},\quad \nu=\sqrt{l^2+m^2}.$$

And since Oξ is at right angles to Oη and Oζ, by § 53 (E), the direction-cosines of Oξ are

$$n\mu-m\nu,\quad l\nu-n\lambda,\quad m\lambda-l\mu;$$

$$i.e.\ \frac{-m}{\sqrt{l^2+m^2}},\quad \frac{l}{\sqrt{l^2+m^2}},\quad 0.$$

Hence we have the scheme:

	x	y	z
ξ	$\frac{-m}{\sqrt{l^2+m^2}}$	$\frac{l}{\sqrt{l^2+m^2}}$	0
η	$\frac{-ln}{\sqrt{l^2+m^2}}$	$\frac{-mn}{\sqrt{l^2+m^2}}$	$\sqrt{l^2+m^2}$
ζ	l	m	n

Ex. 1. Shew that the projection of a conic is a conic of the same species.

The equation $f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a cylinder whose generators are parallel to OZ and pass through the conic $z=0$, $f(x, y)=0$. The equations $lx+my+nz=0$, $f(x, y)=0$ represent the curve in which the plane $lx+my+nz=0$ cuts the cylinder, and the projection of this curve on the plane $z=0$ is the conic. Change the axes as above, and the equations to the curve become

$$\zeta=0,\ a\left(\frac{m\xi+ln\eta}{\sqrt{l^2+m^2}}\right)^2-2h\left(\frac{m\xi+ln\eta}{\sqrt{l^2+m^2}}\right)\left(\frac{l\xi-mn\eta}{\sqrt{l^2+m^2}}\right)+b\left(\frac{l\xi-mn\eta}{\sqrt{l^2+m^2}}\right)^2$$
$$+\ldots=0,$$

and therefore the curve is the conic given by

$$\zeta=0,\ a'\xi^2+2h'\xi\eta+b'\eta^2+\ldots=0,$$

where $a'\equiv\frac{am^2-2hlm+bl^2}{l^2+m^2},\quad h'\equiv\frac{lmn(a-b)-hn(l^2-m^2)}{l^2+m^2},$

$$b'\equiv\frac{n^2(al^2+2hlm+bm^2)}{l^2+m^2}.$$

Whence $h'^2-a'b'=n^2(h^2-ab)$

and $h'^2-a'b'\gtreqless 0$ as $h^2-ab\gtreqless 0.$

Ex. 2. All plane sections of a surface represented by an equation of the second degree are conics.

Take coordinate axes so that a plane section is $z=0$; the equation to the surface is, after transformation, of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

The section by the plane **XOY** is the conic whose equations are

$$z=0, \quad ax^2+2hxy+by^2+2ux+2vy+d=0.$$

The surfaces represented by equations of the second degree are the **conicoids.**

Ex. 3. All parallel plane sections of a conicoid are similar and similarly situated conics.

Take the coordinate plane $z=0$ parallel to a system of parallel plane sections. The equations to the sections by the planes $z=k$, $z=k'$ are then,

$$z=k, \quad ax^2+2hxy+by^2+2x(gk+u)+2y(fk+v)+ck^2+2wk+d=0,$$

$$z=k', \quad ax^2+2hxy+by^2+2x(gk'+u)+2y(fk'+v)+ck'^2+2wk'+d=0.$$

Hence the sections are similar and similarly situated conics.

Ex. 4. Find the conditions that the section of the surface $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ should be (i) a parabola, (ii) an ellipse, (iii) a hyperbola.

(It is sufficient to examine the section by the plane $lx+my+nz=0$, which, by Ex. 3, is a similar conic. The equation to the projection of this section on the plane $z=0$ is obtained by eliminating z between the equations $lx+my+nz=0$, $ax^2+by^2+cz^2=1$, and the projection is a conic of the same species.)

Ans. For a parabola $l^2/a+m^2/b+n^2/c=0$, etc.

Ex. 5. Find the condition that the section of $ax^2+by^2=2z$ by $lx+my+nz=p$ should be a rectangular hyperbola.

(Since rectangular hyperbolas do not, in general, project into rectangular hyperbolas, it will, in this case, be necessary to examine the actual section of the surface by the plane $lx+my+nz=0$ by the method of § 54.) *Ans.* $(a+b)n^2+am^2+bl^2=0$.

Ex. 6. Find the conditions that the section of $ax^2+by^2+cz^2=1$ by $lx+my+nz=p$ should be a circle.

Ans. $l=0$, $m^2(c-a)=n^2(a-b)$; or $m=0$, $n^2(a-b)=l^2(b-c)$; or $n=0$, $l^2(b-c)=m^2(c-a)$.

Ex. 7. If $lx+my=0$ is a circular section of

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{D}xy=1,$$

prove that $$(\mathrm{B}-\mathrm{C})l^2-2\mathrm{D}lm+(\mathrm{A}-\mathrm{C})m^2=0.$$

Ex. 8. Prove that the eccentricity of the section of $xy=z$ by $lx+my+nz=0$, $(l^2+m^2+n^2=1)$, is given by

$$\frac{2}{e^2}=1\pm\frac{lm}{\sqrt{(n^2+l^2)(n^2+m^2)}}.$$

Explain the result when $n=0$.

Ex. 9. Shew that if

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$$

be transformed by change of coordinates from one set of rectangular axes to another with the same origin, the expressions $a+b+c$, $u^2+v^2+w^2$ remain unaltered in value.

Ex. 10. Two sets of rectangular axes through a common origin O meet a sphere whose centre is O in P, Q, R; P′, Q′, R′. Prove that Vol. OPQR′ = ± Vol. OP′Q′R.

Ex. 11. The equations, referred to rectangular axes, of three mutually perpendicular planes, are $p_r-l_rx-m_ry-n_rz=0$, $r=1, 2, 3$. Prove that if (ξ, η, ζ) is at a distance d from each of them,

$$\frac{\xi-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{\eta-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}$$
$$=\frac{\zeta-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}=d.$$

Ex. 12. If the axes of x, y, z are rectangular, prove that the substitutions

$$x=\frac{\xi}{\sqrt{3}}+\frac{\eta}{\sqrt{2}}+\frac{\zeta}{\sqrt{6}},\quad y=\frac{\xi}{\sqrt{3}}-\frac{2\zeta}{\sqrt{6}},\quad z=\frac{\xi}{\sqrt{3}}-\frac{\eta}{\sqrt{2}}+\frac{\zeta}{\sqrt{6}}$$

give a transformation to another set of rectangular axes in which the plane $x+y+z=0$ becomes the plane $\xi=0$, and hence prove that the section of the surface $yz+zx+xy+a^2=0$ by the plane $x+y+z=0$ is a circle of radius $\sqrt{2}\,.\,a$.

***55.** If OX, OY, OZ are rectangular axes, and Oξ, Oη, Oζ are oblique axes whose direction-cosines, referred to OX, OY, OZ, are l_1, m_1, n_1; l_2, m_2, n_2, l_3, m_3, n_3, then projecting on OX, OY, OZ; Oξ, Oη, Oζ, as in § 52, we obtain

$$\left.\begin{aligned} x&=l_1\xi+l_2\eta+l_3\zeta,\\ y&=m_1\xi+m_2\eta+m_3\zeta,\\ z&=n_1\xi+n_2\eta+n_3\zeta.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(\text{A})$$

$$\left.\begin{aligned} \xi+\eta\cos\nu+\zeta\cos\mu&=l_1x+m_1y+n_1z,\\ \xi\cos\nu+\eta+\zeta\cos\lambda&=l_2x+m_2y+n_2z,\\ \xi\cos\mu+\eta\cos\lambda+\zeta&=l_3x+m_3y+n_3z,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\text{B})$$

where the angles ηOζ, ζOξ, ξOη are λ, μ, ν. The equations (B) can also be deduced from (A) by multiplying in turn by l_1, m_1, n_1, etc., and adding. Again, from (A),

$$\xi=\begin{vmatrix} x, & y, & z\\ l_2, & m_2, & n_2\\ l_3, & m_3, & n_3\end{vmatrix}\div\begin{vmatrix} l_1, & m_1, & n_1\\ l_2, & m_2, & n_2\\ l_3 & m_3, & n_3\end{vmatrix}, \text{ etc., etc. }\ldots(\text{C})$$

By means of (A) and (C) we can transform from rectangular to oblique axes and *vice versa.*

Cor. Since x, y, z are linear functions of ξ, η, ζ and *vice versa,* the degree of any equation is unaltered by transformation from rectangular to oblique axes or from oblique to rectangular axes. The transformation from one set of oblique axes to another can be performed, by introducing a set of rectangular axes, in the above two steps, and hence in this most general case the degree of the equation is unaltered by the transformation.

Ex. 1. The equation $x^2+4(y^2+z^2)=2$ is transformed by change from rectangular axes, the new axes being oblique, and having direction-cosines proportional to

$$2,\ 1,\ 1\ ;\quad 4,\ \sqrt{3}-1,\ -\sqrt{3}-1\ ;\quad 4,\ -\sqrt{3}-1,\ \sqrt{3}-1.$$

Shew that the new equation is $x^2+y^2+z^2=1$.

Ex. 2. If P, Q, R are $(\xi_r,\ \eta_r,\ \zeta_r)$, $r=1,\ 2,\ 3$, referred to a set of oblique axes through an origin O, prove that

$$6\,.\,\text{Vol. OPQR}=\begin{vmatrix}\xi_1, & \eta_1, & \zeta_1\\ \xi_2, & \eta_2, & \zeta_2\\ \xi_3, & \eta_3, & \zeta_3\end{vmatrix}\cdot\begin{vmatrix}1, & \cos\nu, & \cos\mu\\ \cos\nu, & 1, & \cos\lambda\\ \cos\mu, & \cos\lambda, & 1\end{vmatrix}^{\frac{1}{2}}.$$

(Use § 55 (B) ; cf. § 51, Ex. 9.)

*Examples I.

1. The gnomon of a sundial is in the meridian at an elevation λ (equal to the latitude), and the sun is due east at an elevation α. Find the angle θ that the shadow makes with the N. and S. line of the dial.

2. Find the equations to the line through (1, 1, 1) which meets both the lines $\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-2}{4}$, $x=2y=3z$, and shew that its intersection with the second line is $\left(\dfrac{15}{26}, \dfrac{15}{52}, \dfrac{5}{26}\right)$.

3. If OA, OB, OC have direction-ratios l_r, m_r, n_r, $r=1,\ 2,\ 3$; and OA′, OB′, OC′ bisect the angles BOC, COA, AOB, the planes AOA′, BOB′, COC′ pass through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3}.$$

4. P is a given point and PM, PN are the perpendiculars from P to the planes ZOX, XOY. OP makes angles θ, α, β, γ with the planes OMN and the (rectangular) coordinate planes. Prove that

$$\operatorname{cosec}^2\theta=\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma.$$

5. Shew that the locus of lines which meet the lines

$$\frac{x \pm a}{0} = \frac{y}{\sin \alpha} = \frac{z}{\mp \cos \alpha}$$

at the same angle is

$$(xy \cos \alpha - az \sin \alpha)(xz \sin \alpha - ay \cos \alpha) = 0.$$

6. Find the locus of a straight line which meets **OX** and the circle $x^2+y^2=c^2$, $z=h$, so that the distance between the points of section is $\sqrt{c^2+h^2}$.

7. If three rectangular axes be rotated about the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ into new positions, and the direction-cosines of the new axes referred to the old are l_1, m_1, n_1, etc.; then if

$$l_1 = +(m_2 n_3 - m_3 n_2), \quad \lambda(m_3+n_2) = \mu(n_1+l_3) = \nu(l_2+m_1);$$

also if θ is the angle through which the system is rotated,

$$\sin^2\frac{\theta}{2} = \frac{1-l_1}{2} \cdot \frac{\lambda^2+\mu^2+\nu^2}{\mu^2+\nu^2}.$$

8. If the shortest distances between lines 1, 2, 3 are parallel to lines 4, 5, 6, then the shortest distances between the lines 4, 5, 6 are parallel to the lines 1, 2, 3.

9. Any three non-intersecting lines can be made the edges of a parallelepiped, and if the lines are $\frac{x-\alpha_r}{l_r}=\frac{y-\beta_r}{m_r}=\frac{z-\gamma_r}{n_r}$, $r=1, 2, 3$, the lengths of the edges are

$$\begin{vmatrix} \alpha_2-\alpha_3, & \beta_2-\beta_3, & \gamma_2-\gamma_3 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \div \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}, \text{ etc.}$$

Consider the case where the denominator is zero.

10. **OA, OB, OC** are edges of a parallelepiped and **R** is the corner opposite to **O**. **OP** and **RQ** are perpendiculars to the plane **ABC**. Compare the lengths of **OP** and **RQ**. If the figure is rectangular and **O** is taken as origin, and the plane **ABC** is given by $lx+my+nz=p$, **PQ** has direction-cosines proportional to $l^{-1}-3l$, $m^{-1}-3m$, $n^{-1}-3n$, and $\mathrm{PQ}^2=\mathrm{OR}^2-9\,.\,\mathrm{OP}^2$.

11. **OS** is the diagonal of the cube of which **OP, OQ, OR** are edges. **OU** is the diagonal of the parallelepiped of which **OQ, OR, OS** are edges, and **OV** and **OW** are formed similarly. Find the coordinates of **U, V, W**, and if **OT** is the diagonal of the parallelepiped of which **OU, OV, OW** are edges, shew that **OT** coincides with **OS** and that $\mathrm{OT}=5\,.\,\mathrm{OS}$.

12. Find the equations to the straight line through the origin which meets at right angles the line whose equations are

$$(b+c)x+(c+a)y+(a+b)z=k=(b-c)x+(c-a)y+(a-b)z,$$

and find the coordinates of the point of section.

13. Find the locus of a point which moves so that the ratio of its distances from two given lines is constant.

14. A line is parallel to the plane $y+z=0$ and intersects the circles $x^2+y^2=a^2$, $z=0$; $x^2+z^2=a^2$, $y=0$; find the surface it generates.

15. Find the equation to the surface generated by a straight line which is parallel to the line $y=mx$, $z=nx$, and intersects the ellipse

$$x^2/a^2+y^2/b^2=1, \quad z=0.$$

16. A plane triangle, sides a, b, c, is placed so that the mid-points of the sides are on the axes (rectangular). Shew that the lengths intercepted on the axes are given by

$$l^2=(b^2+c^2-a^2)/8, \quad m^2=(c^2+a^2-b^2)/8, \quad n^2=(a^2+b^2-c^2)/8,$$

and that the coordinates of the vertices are $(-l, m, n)$, $(l, -m, n)$, $(l, m, -n)$.

17. Lines are drawn to meet two given lines and touch the right circular cylinder whose axis is the S.D. (length $2c$), and radius c. Find the surface generated.

18. The section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ is a parabola of latus rectum $2\mathsf{L}$. Prove that

$$\mathsf{L}(l^2/a^2+m^2/b^2+n^2/c^2)^{\frac{3}{2}}=p(l^2+m^2+n^2)/abc.$$

19. A line moves so as to intersect the line $z=0$, $x=y$; and the circles $x=0$, $y^2+z^2=r^2$; $y=0$, $z^2+x^2=r^2$. Prove that the equation to the locus is
$$(x+y)^2\{z^2+(x-y)^2\}=r^2(x-y)^2.$$

20. Prove that $\dfrac{a}{y-z}+\dfrac{b}{z-x}+\dfrac{c}{x-y}=0$ represents a pair of planes whose line of intersection is equally inclined to the axes.

21. Find the surface generated by a straight line which revolves about a given straight line at a constant distance from it and makes a given angle with it.

22. Shew that $x^2+y^2+z^2-3xy-3zx-3yz=1$ represents a surface of revolution about the line $x=y=z$, and find the equations to the generating curve.

23. L_1, L_2, L_3 are three given straight lines and the directions of L_1 and L_2 are at right angles. Find the locus of the line joining the feet of the perpendiculars from any point on L_3 to L_1 and L_2.

24. The ends of diameters of the ellipse $z=c$, $x^2/a^2+y^2/b^2=1$ are joined to the corresponding ends of the conjugates of parallel diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$. Find the equation to the surface generated by the joining lines.

25. A and B are two points on a given plane and AP, BQ are two lines in given directions at right angles to AB. Shew that for all lines PQ, parallel to the plane, $\mathsf{AP} : \mathsf{BQ}$ is constant, and that all such lines lie on a conicoid.

26. The vertex A of a triangle ABC lies on a given line; AB and AC pass through given points; B and C lie on given planes; shew that the locus of BC is a conicoid.

27. Prove that the equation to the two planes inclined at an angle α to the xy-plane and containing the line $y=0$, $z\cos\beta=x\sin\beta$, is

$$(x^2+y^2)\tan^2\beta+z^2-2zx\tan\beta=y^2\tan^2\alpha.$$

28. A line moves so as to meet the lines $\dfrac{x}{\cos\alpha}=\dfrac{y}{\pm\sin\alpha}=\dfrac{z\mp c}{0}$ in **A** and **B** and pass through the curve $yz=k^2$, $x=0$. Prove that the locus of the mid-point of **AB** is a curve of the third degree, two of whose asymptotes are parallel to the given lines.

29. Given two non-intersecting lines whose directions are at right angles and whose S.D. is **AB**, and a circle whose centre **C** is on **AB** and plane parallel to the lines. Shew that the locus of a variable line which intersects the given lines and circle is a surface whose sections by planes parallel to the lines are ellipses whose centres lie on **AB**, and that the section by the plane through **C′**, another point of **AB**, is a circle, if **C**, **C′** are harmonic conjugates with respect to **A** and **B**.

30. If the axes are rectangular the locus of the centre of a circle of radius a which always intersects them is

$$x\sqrt{a^2-y^2-z^2}+y\sqrt{a^2-z^2-x^2}+z\sqrt{a^2-x^2-y^2}=a^2.$$

31. A line is drawn to meet $y=x\tan\alpha$, $z=c$; $y=-x\tan\alpha$, $z=-c$, so that the length intercepted on it is constant. Shew that its equations may be written in the form

$$\frac{x-k\sin\theta\cot\alpha}{k\cos\theta}=\frac{y-k\cos\theta\tan\alpha}{k\sin\theta}=\frac{z}{c},$$

where k is a constant and θ a parameter. Deduce the equation to the locus of the line.

32. Find the equation to the surface generated by a straight line which is parallel to the plane $z=0$ and intersects the line $x=y=z$, and the curve $x+2y=4z$, $x^2+y^2=a^2$.

33. Through a fixed line **L**, which lies in the xy-plane but does not pass through the origin, is drawn a plane which intersects the planes $x=0$ and $y=0$ in lines **M** and **N** respectively. Through **M** and a fixed point **A**, and through **N** and another fixed point **B**, planes are drawn. Find the locus of their line of intersection.

34. The axes are rectangular and a point **P** moves on the fixed plane $x/a+y/b+z/c=1$. The plane through **P** perpendicular to **OP** meets the axes in **A**, **B**, **C**. The planes through **A**, **B**, **C** parallel to **YOZ**, **ZOX**, **XOY** intersect in **Q**. Shew that the locus of **Q** is $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{ax}+\dfrac{1}{by}+\dfrac{1}{cz}$.

35. **AB** and **CD** are given non-intersecting lines. Any plane through **AB** cuts **CD** in **P**, and **PQ** is normal to it at **P**. Find the locus of **PQ**.

36. Find the equation to a plane which touches each of the circles $x=0$, $y^2+z^2=a^2$; $y=0$, $z^2+x^2=b^2$; $z=0$, $x^2+y^2=c^2$. How many such planes are there?

37. Find the locus of the position of the eye at which two given non-intersecting lines appear to cut at right angles.

38. Four given points of a variable line lie on the faces of a quadrilateral prism. Shew that any other point of the line describes a line which is parallel to the edges of the prism.

39. The locus of the harmonic conjugates of P with respect to the two points in which any secant through P cuts a pair of planes is the polar of P with respect to the planes. Prove that the equation to the polar of (x_1, y_1, z_1) with respect to $u=0$, $v=0$, is $\frac{u}{u_1}+\frac{v}{v_1}=0$, where u_1 is the result of substituting x_1, y_1, z_1 for x, y, z in u, etc. Shew also that the polars of P with respect to the pairs of planes that form a trihedral angle cut those planes in three coplanar lines.

40. Any line meets the faces BCD, CDA, DAB, ABC of a tetrahedron ABCD in A′, B′, C′, D′. Prove that the mid-points of AA′, BB′, CC′, DD′ are coplanar.

41. If the axes are rectangular, and λ, μ, ν are the angles between the lines of intersection of the planes $a_r x+b_r y+c_r z=0$, $r=1, 2, 3$, prove that

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \frac{(\Sigma a_1^2 . \Sigma a_2^2 . \Sigma a_3^2)^{\frac{1}{2}}(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)}{\sin\lambda\sin\mu\sin\nu}.$$

42. The equations $x=\lambda z+\mu$, $y=(\lambda^3-2\lambda\mu)z+\mu(\lambda^2-\mu)$, where λ and μ are parameters, determine a system of lines. Find the locus of those which intersect the z-axis. Prove that two lines of the system pass through any given point unless the given point lies on a certain curve, when an infinite number of lines pass through it, and find the equations to the curve.

CHAPTER V.

THE SPHERE.

56. Equation to a sphere. If the axes are rectangular the square of the distance between the points P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) is given by $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$, and therefore the equation to the sphere whose centre is P and whose radius is of length r, is

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2.$$

Any equation of the form

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$$

can be written

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2};$$

and therefore represents a sphere whose centre is

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right) \text{ and radius } \frac{\sqrt{u^2+v^2+w^2-ad}}{a}.$$

Ex. 1. Find the equation to the sphere whose centre is $(2, -3, 4)$ and radius 5. *Ans.* $x^2+y^2+z^2-4x+6y-8z+4=0$.

Ex. 2. Find the centre and radius of the sphere given by

$x^2+y^2+z^2-2x+4y-6z=11$. *Ans.* $(1, -2, 3)$, 5.

Ex. 3. Shew that the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

represents the sphere on the join of (x_1, y_1, z_1), (x_2, y_2, z_2) as diameter.

Ex. 4. Find the equation to the sphere through the points

$$(0, 0, 0), \quad (0, 1, -1), \quad (-1, 2, 0), \quad (1, 2, 3).$$

Ans. $7(x^2+y^2+z^2)-15x-25y-11z=0$.

Ex. 5. Find the equation to the sphere which passes through the point (α, β, γ) and the circle $z=0$, $x^2+y^2=a^2$.

Ans. $\gamma(x^2+y^2+z^2-a^2)=z(\alpha^2+\beta^2+\gamma^2-a^2)$.

Ex. 6. Find the equations to the spheres through the circle

$$x^2+y^2+z^2=9, \quad 2x+3y+4z=5;$$

and (i) the origin, (ii) the point (1, 2, 3).

Ans. (i) $5(x^2+y^2+z^2)-18x-27y-36z=0$;
(ii) $3(x^2+y^2+z^2)-2x-3y-4z-22=0$.

Ex. 7. The plane ABC, whose equation is $x/a+y/b+z/c=1$, meets the axes in A, B, C. Find equations to determine the circumcircle of the triangle ABC, and obtain the coordinates of its centre.

Ans. $x/a+y/b+z/c=1$, $x^2+y^2+z^2-ax-by-cz=0$;

$$\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}, \quad \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}, \quad \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}.$$

***Ex. 8.** If the axes are oblique, find the equation to the sphere whose centre is (x_1, y_1, z_1), and radius r.

Ans. $\Sigma(x-x_1)^2+2\Sigma(y-y_1)(z-z_1)\cos\lambda=r^2$.

***Ex. 9.** Prove that the necessary and sufficient conditions that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

referred to oblique axes, should represent a sphere, are

$$a=b=c=\frac{f}{\cos\lambda}=\frac{g}{\cos\mu}=\frac{h}{\cos\nu}$$

Prove that the radius is $\frac{1}{a^2}\left(\frac{-S}{\Delta}\right)^{\frac{1}{2}}$, where

$$S\equiv\begin{vmatrix} a, & a\cos\nu, & a\cos\mu, & u \\ a\cos\nu, & a, & a\cos\lambda, & v \\ a\cos\mu, & a\cos\lambda, & a, & w \\ u, & v, & w, & d \end{vmatrix} \text{ and } \Delta\equiv\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}.$$

57. Tangents and tangent planes. If P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) are points on the sphere $x^2+y^2+z^2=a^2$, then

$$x_1^2+y_1^2+z_1^2=a^2=x_2^2+y_2^2+z_2^2,$$

and therefore

$$(x_1-x_2)(x_1+x_2)+(y_1-y_2)(y_1+y_2)+(z_1-z_2)(z_1+z_2)=0.$$

Now the direction-cosines of PQ are proportional to x_1-x_2, y_1-y_2, z_1-z_2; and if M is the mid-point of PQ and O is the origin, the direction-cosines of OM are proportional to x_1+x_2, y_1+y_2, z_1+z_2. Therefore PQ is at right angles to **OM**. Suppose that **OM** meets the sphere in **A** and that **PQ** moves parallel to itself with its mid-point, **M**, on **OA**. Then when **M** is at **A**, **PQ** is a **tangent** to the sphere at **A**, and hence a tangent at **A** is at right angles to **OA**, and the locus of the tangents at **A** is the plane through **A** at right angles to **OA**. This plane is the **tangent plane** at **A**. The equation to the tangent plane at **A**, (α, β, γ), is

$$(x-\alpha)\alpha+(y-\beta)\beta+(z-\gamma)\gamma=0,$$

or

$$x\alpha+y\beta+z\gamma=\alpha^2+\beta^2+\gamma^2=a^2.$$

Ex. 1. Find the equation to the tangent plane at

$$(a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$

to the sphere $x^2+y^2+z^2=a^2$.

Ans. $x\cos\theta\sin\phi+y\sin\theta\sin\phi+z\cos\phi=a$.

Ex. 2. Find the equation to the tangent plane at (x', y', z') to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Ans. $xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d=0$.

Ex. 3. Find the condition that the plane $lx+my+nz=p$ should touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Ans. $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

Ex. 4. Find the equations to the spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$, and touch the plane $4x+3y=15$.

Ans. $x^2+y^2+z^2+2x+4y+6z-11=0$,

$5x^2+5y^2+5z^2-4x-8y-12z-13=0$.

Ex. 5. Prove that the tangent planes to the spheres

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

at any common point are at right angles if

$$2uu_1+2vv_1+2ww_1=d+d_1.$$

***58. Radical plane of two spheres.** *If any secant through a given point* **O** *meets a given sphere in* **P** *and* **Q**, **OP . OQ** *is constant.*

The equations to the line through **O**, (α, β, γ), whose direction-cosines are l, m, n, are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\quad(=r).$$

The point on this line, whose distance from O is r, has coordinates $\alpha+lr$, $\beta+mr$, $\gamma+nr$, and lies on the sphere

$$\mathsf{F}(xyz) \equiv a(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

if $$ar^2+r\left(l\frac{\partial \mathsf{F}}{\partial \alpha}+m\frac{\partial \mathsf{F}}{\partial \beta}+n\frac{\partial \mathsf{F}}{\partial \gamma}\right)+\mathsf{F}(\alpha, \beta, \gamma)=0.$$

This equation gives the lengths of OP and OQ, and hence $\mathsf{OP}.\mathsf{OQ}$ is given by $\mathsf{F}(\alpha, \beta, \gamma)/a$, which is the same for all secants through O.

Definition. The measure of $\mathsf{OP}.\mathsf{OQ}$ is **the power** of O with respect to the sphere.

If $$\mathsf{S}_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$\mathsf{S}_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

are the equations to two spheres, the locus of points whose powers with respect to the spheres are equal is the plane given by

$$\mathsf{S}_1=\mathsf{S}_2, \text{ or } 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0.$$

This plane is called the **radical plane** of the two spheres. It is evidently at right angles to the line joining the centres.

The radical planes of three spheres taken two by two pass through one line.

(The equations to the line are $\mathsf{S}_1=\mathsf{S}_2=\mathsf{S}_3$.)

The radical planes of four spheres taken two by two pass through one point.

(The point is given by $\mathsf{S}_1=\mathsf{S}_2=\mathsf{S}_3=\mathsf{S}_4$.)

The equations to any two spheres can be put in the form

$$x^2+y^2+z^2+2\lambda_1x+d=0, \quad x^2+y^2+z^2+2\lambda_2x+d=0.$$

(Take the line joining the centres as x-axis and the radical plane as $x=0$.)

The equation $x^2+y^2+z^2+2\lambda x+d=0$, where λ is a parameter, represents a system of spheres any two of which have the same radical plane. The spheres are said to be *coaxal.*

Ex. 1. Prove that the members of the coaxal system intersect one another, touch one another, or do not intersect one another, according as $d \lesseqgtr 0$.

Ex. 2. Shew that the centres of the two spheres of the system which have zero-radius are at the points $(\pm\sqrt{d}, 0, 0)$. (These are the *limiting-points* of the system.)

Ex. 3. Shew that the equation $x^2+y^2+z^2+2\mu y+2\nu z-d=0$, where μ and ν are parameters, represents a system of spheres passing through the limiting points of the system $x^2+y^2+z^2+2\lambda x+d=0$, and cutting every member of that system at right angles.

Ex. 4. The locus of points whose powers with respect to two given spheres are in a constant ratio is a sphere coaxal with the two given spheres.

Ex. 5. Shew that the spheres which cut two given spheres along great circles all pass through two fixed points.

*Examples II.

1. A sphere of constant radius r passes through the origin, O, and cuts the axes (rectangular) in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2.$$

2. P is a variable point on a given line and A, B, C are its projections on the axes. Shew that the sphere OABC passes through a fixed circle.

3. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Shew that the locus of the centre of the sphere OABC is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

4. If the three diagonals of an octahedron intersect at right angles, the feet of the perpendiculars from the point of intersection to the faces of the octahedron lie on a sphere. If a, α; b, β; c, γ are the measures of the segments of the diagonals, the centre (ξ, η, ζ) of the sphere is given by

$$\frac{2\xi}{a^{-1}+\alpha^{-1}}=\frac{2\eta}{b^{-1}+\beta^{-1}}=\frac{2\zeta}{c^{-1}+\gamma^{-1}}=\frac{1}{(a\alpha)^{-1}+(b\beta)^{-1}+(c\gamma)^{-1}},$$

the diagonals being taken as coordinate axes. Prove that the points where the perpendiculars meet the opposite faces also lie on the sphere.

5. Prove that the locus of the centres of spheres which pass through a given point and touch a given plane is a conicoid.

6. Find the locus of the centres of spheres that pass through a given point and intercept a fixed length on a given straight line.

7. Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.

8. Prove that the centres of spheres which touch the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$, lie upon the conicoid $mxy+cz(1+m^2)=0$.

9. If the opposite edges of a tetrahedron are at right angles the centre of gravity is the mid-point of the line joining the point of concurrence of the perpendiculars and the centre of the circumscribing sphere.

10. If the opposite edges of a tetrahedron are at right angles the mid-points of the edges and the feet of the perpendiculars from the vertices to the edges lie upon a sphere whose centre is the centre of gravity of the tetrahedron.

11. The sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant.

12. With any point **P** of a given plane as centre a sphere is described whose radius is equal to the tangent from **P** to a given sphere. Prove that all such spheres pass through two fixed points.

13. If $\lambda=\mu=\nu=\pi/3$, the plane and surface given by

$$x+y+z=0, \quad yz+zx+xy+a^2=0,$$

intersect in a circle of radius a.

14. If r is the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=0,$$

prove that

$$(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2.$$

15. Prove that the equations to the spheres that pass through the points $(4, 1, 0)$, $(2, -3, 4)$, $(1, 0, 0)$, and touch the plane $2x+2y-z=11$, are

$$x^2+y^2+z^2-6x+2y-4z+5=0,$$

$$16x^2+16y^2+16z^2-102x+50y-49z+86=0.$$

16. Prove that the equation to a sphere, which lies in the octant **OXYZ** and touches the coordinate planes, is of the form

$$x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0.$$

Prove that in general two spheres can be drawn through a given point to touch the coordinate planes, and find for what positions of the point the spheres are (i) real; (ii) coincident.

17. **A** is a point on **OX** and **B** on **OY** so that the angle **OAB** is constant $(=\alpha)$. On **AB** as diameter a circle is described whose plane is parallel to **OZ**. Prove that as **AB** varies the circle generates the cone

$$2xy-z^2 \sin 2\alpha=0.$$

18. **POP′** is a variable diameter of the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$, and a circle is described in the plane **PP′ZZ′** on **PP′** as diameter. Prove that as **PP′** varies, the circle generates the surface

$$(x^2+y^2+z^2)(x^2/a^2+y^2/b^2)=x^2+y^2.$$

19. Prove that the equation to the sphere circumscribing the tetrahedron whose sides are

$$\frac{y}{b}+\frac{z}{c}=0,\quad \frac{z}{c}+\frac{x}{a}=0,\quad \frac{x}{a}+\frac{y}{b}=0,\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

is

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0.$$

20. A variable plane is parallel to the given plane $x/a+y/b+z/c=0$, and meets the axes in A, B, C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

21. Find the locus of the centre of a variable sphere which passes through the origin O and meets the axes in A, B, C, so that the volume of the tetrahedron OABC is constant.

22. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

23. The tangents drawn from a point P to a sphere are all equal to the distance of P from a fixed tangent plane to the sphere. Prove that the locus of P is a paraboloid of revolution.

24. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\quad 5y+6z+1=0\,;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\quad x+2y-7z=0\,;$$

lie on the same sphere, and find its equation.

25. Find the conditions that the circles

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,\quad lx+my+nz=p\,;$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0,\quad l'x+m'y+n'z=p'\,;$$

should lie on the same sphere.

26. OA, OB, OC are mutually perpendicular lines through the origin, and their direction-cosines are l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3. If OA$=a$, OB$=b$, OC$=c$, prove that the equation to the sphere OABC is

$$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)-z(an_1+bn_2+cn_3)=0.$$

CHAPTER VI.

THE CONE.

59. Equation to a cone. A cone is a surface generated by a straight line which passes through a fixed point and intersects a given curve. If the given point O, say, be chosen as origin, the equation to the cone is homogeneous. For if P, (x', y', z') is any point on the cone, x', y', z' satisfy the equation. And since any point on OP is on the cone, and has coordinates (kx', ky', kz'), the equation is also satisfied by kx', ky', kz' for all values of k, and therefore must be homogeneous.

Cor. If $x/l=y/m=z/n$ is a generator of the cone represented by the homogeneous equation $f(x, y, z)=0$, then $f(l, m, n)=0$. Conversely, if the direction-ratios of a straight line which always passes through a fixed point satisfy a homogeneous equation, the line is a generator of a cone whose vertex is at the point.

Ex. 1. The line $x/l=y/m=z/n$, where $2l^2+3m^2-5n^2=0$, is a generator of the cone $2x^2+3y^2-5z^2=0$.

Ex. 2. Lines drawn through the point (α, β, γ) whose direction-ratios satisfy $al^2+bm^2+cn^2=0$ generate the cone

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0.$$

Ex. 3. Shew that the equation to the right circular cone whose vertex is O, axis OZ, and semi-vertical angle α, is $x^2+y^2=z^2\tan^2\alpha$.

Ex. 4. The general equation to the cone of the second degree which passes through the axes is $fyz+gzx+hxy=0$.

The general equation to the cone of the second degree is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

and this is to be satisfied by the direction-ratios of the axes, *i.e.* by 1, 0, 0; 0, 1, 0; 0, 0, 1.

Ex. 5. A cone of the second degree can be found to pass through any five concurrent lines.

Ex. 6. A cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.

Take one set as coordinate axes, and let the direction-cosines of the others be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. The equation to a cone containing the coordinate axes is $fyz+gzx+hxy=0$. If this cone also contains the first two axes of the second set,

$$fm_1n_1+gn_1l_1+hl_1m_1=0,$$

$$fm_2n_2+gn_2l_2+hl_2m_2=0.$$

Therefore, since $m_1n_1+m_2n_2+m_3n_3=0$, etc.,

$$fm_3n_3+gn_3l_3+hl_3m_3=0\,;$$

so that the cone contains the remaining axis.

Ex. 7. The equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx+my+nz=p$ and the surface $ax^2+by^2+cz^2=1$ is

$$ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{p}\right)^2.$$

Ex. 8. Find the equations to the cones with vertex at the origin which pass through the curves given by

(i) $x^2+y^2+z^2+2ax+b=0, \quad lx+my+nz=p$;

(ii) $ax^2+by^2=2z, \quad lx+my+nz=p$;

(iii) $x^2/a^2+y^2/b^2+z^2/c^2=1, \quad x^2/\alpha^2+y^2/\beta^2=2z$.

Ans. (i) $(x^2+y^2+z^2)p^2+2apx(lx+my+nz)+b(lx+my+nz)^2=0$;

(ii) $(ax^2+by^2)p=2z(lx+my+nz)$;

(iii) $4z^2(x^2/a^2+y^2/b^2+z^2/c^2)=(x^2/\alpha^2+y^2/\beta^2)^2$.

Ex. 9. The plane $x/a+y/b+z/c=1$ meets the coordinate axes in **A, B, C**. Prove that the equation to the cone generated by lines drawn from **O** to meet the circle **ABC** is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

Ex. 10. Find the equation to the cone whose vertex is the origin and base the circle, $x=a$, $y^2+z^2=b^2$, and shew that the section of the cone by a plane parallel to the plane **XOY** is a hyperbola.

Ans. $a^2(y^2+z^2)=b^2x^2$.

Ex. 11. Shew that the equation to the cone whose vertex is the origin and base the curve $z=k$, $f(x, y)=0$ is $f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0$.

60. Angle between lines in which a plane cuts a cone. We find it convenient to introduce here the following notation, to which we shall adhere throughout the book.

$$\mathrm{D} \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

$$\mathrm{A} \equiv \frac{\partial \mathrm{D}}{\partial a} = bc - f^2, \qquad \mathrm{B} \equiv \frac{\partial \mathrm{D}}{\partial b} = ca - g^2, \qquad \mathrm{C} \equiv \frac{\partial \mathrm{D}}{\partial c} = ab - h^2;$$

$$\mathrm{F} \equiv \frac{1}{2}\frac{\partial \mathrm{D}}{\partial f} = gh - af, \quad \mathrm{G} \equiv \frac{1}{2}\frac{\partial \mathrm{D}}{\partial g} = hf - bg, \quad \mathrm{H} \equiv \frac{1}{2}\frac{\partial \mathrm{D}}{\partial h} = fg - ch.$$

The student can easily verify that

$$\mathrm{BC} - \mathrm{F}^2 = a\mathrm{D}, \quad \mathrm{CA} - \mathrm{G}^2 = b\mathrm{D}, \quad \mathrm{AB} - \mathrm{H}^2 = c\mathrm{D};$$

$$\mathrm{GH} - \mathrm{AF} = f\mathrm{D}, \quad \mathrm{HF} - \mathrm{BG} = g\mathrm{D}, \quad \mathrm{FG} - \mathrm{CH} = h\mathrm{D}.$$

In what follows we use P^2 to denote

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & 0 \end{vmatrix},$$

or $$-(\mathrm{A}u^2 + \mathrm{B}v^2 + \mathrm{C}w^2 + 2\mathrm{F}vw + 2\mathrm{G}wu + 2\mathrm{H}uv).$$

The axes being rectangular to find the angle between the lines in which the plane $ux + vy + wz = 0$ *cuts the cone*

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

If the line $x/l = y/m = z/n$ lies in the plane,

$$ul + vm + wn = 0; \quad \text{.....................(1)}$$

if it lies on the cone,

$$f(l, m, n) = 0. \quad \text{.......................(2)}$$

Eliminate n between (1) and (2), and we obtain

$$l^2(cu^2 + aw^2 - 2gwu) + 2lm(hw^2 + cuv - fuw - gvw)$$
$$+ m^2(cv^2 + bw^2 - 2fvw) = 0. \quad \text{...........(3)}$$

Now the direction-cosines of the two lines of section satisfy the equations (1) and (2), and therefore they satisfy equation (3). Therefore if they are l_1, m_1, n_1; l_2, m_2, n_2;

$$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}$$

$$=\frac{l_1m_2+l_2m_1}{-2(hw^2+cuv-fuw-gvw)}$$

$$=\frac{l_1m_2-l_2m_1}{\pm 2\{(hw^2+cuv\ldots)^2-(bw^2\ldots)(cu^2\ldots)\}^{\frac{1}{2}}} \qquad\ldots\ldots\;\ldots\ldots(4)$$

$$=\frac{l_1m_2-l_2m_1}{\pm 2w\mathsf{P}}.$$

From the symmetry, each of the expressions in (4) is seen to be equal to

$$\frac{n_1n_2}{av^2+bu^2-2huv}=\frac{m_1n_2-m_2n_1}{\pm 2u\mathsf{P}}=\frac{n_1l_2-n_2l_1}{\pm 2v\mathsf{P}}$$

But if θ is the angle between the lines,

$$\frac{\cos\theta}{l_1l_2+m_1m_2+n_1n_2}=\frac{\sin\theta}{\{\Sigma(m_1n_2-m_2n_1)^2\}^{\frac{1}{2}}}.$$

$$\therefore \frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}=\frac{\sin\theta}{\pm 2(u^2+v^2+w^2)^{\frac{1}{2}}\mathsf{P}}.$$

Ex. 1. Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$.

Ans. $\dfrac{x}{-1}=\dfrac{y}{4}=\dfrac{z}{2}$; $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{0}$.

Ex. 2. Find the angles between the lines of section of the following planes and cones :

(i) $6x-10y-7z=0$, $108x^2-20y^2-7z^2=0$;

(ii) $3x+y+5z=0$, $6yz-2zx+5xy=0$;

(iii) $2x-3y+z=0$, $3x^2-5y^2-7z^2+36yz-20zx-2xy=0$.

Ans. (i) $\cos^{-1}\dfrac{16}{21}$, (ii) $\cos^{-1}\dfrac{1}{6}$, (iii) $\cos^{-1}\dfrac{5}{\sqrt{39}}$.

Ex. 3. Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

61. Condition of tangency of plane and cone. If $\mathrm{P}=0$, or

$$\mathrm{A}u^2+\mathrm{B}v^2+\mathrm{C}w^2+2\mathrm{F}vw+2\mathrm{G}wu+2\mathrm{H}uv=0, \quad \text{......(1)}$$

then $\sin\theta=0$, and therefore the lines of section coincide, or the plane touches the cone. Equation (1) shews that the line $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$, *i.e.* the normal through O to the plane, is a generator of the cone

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{F}yz+2\mathrm{G}zx+2\mathrm{H}xy=0. \quad \text{.........(2)}$$

Similarly, since we have $\mathrm{BC}-\mathrm{F}^2=a\mathrm{D}$, and the corresponding equations at the head of paragraph **60**, it follows that a normal through the origin to a tangent plane to the cone (2) is a generator of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

i.e. of the given cone. The two cones are therefore such that each is the locus of the normals drawn through the origin to the tangent planes to the other, and they are on that account said to be *reciprocal.*

Ex. 1. Prove that the cones $ax^2+by^2+cz^2=0$ and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$ are reciprocal.

Ex. 2. Prove that tangent planes to the cone $lyz+mzx+nxy=0$ are at right angles to generators of the cone

$$l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0.$$

Ex. 3. Prove that perpendiculars drawn from the origin to tangent planes to the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0$$

lie on the cone $19x^2+11y^2+3z^2+6yz-10zx-26xy=0$.

Ex. 4. Shew that the general equation to a cone which touches the coordinate planes is $a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0$.

62. Condition that the cone has three mutually perpendicular generators. The condition that the plane should cut the cone in perpendicular generators is

$$(a+b+c)(u^2+v^2+w^2)=f(u,\ v,\ w). \quad \text{............(1)}$$

If also the normal to the plane lies on the cone, we have

$$f(u,\ v,\ w)=0,$$

and therefore

$$a+b+c=0.$$

In this case the cone has three mutually perpendicular generators, viz., the normal to the plane and the two perpendicular lines in which the plane cuts the cone.

If $a+b+c=0$, the cone has an infinite number of sets of mutually perpendicular generators. For if $ux+vy+wz=0$ be any plane whose normal lies on the cone, then

$$f(u, v, w)=0,$$

and therefore $(a+b+c)(u^2+v^2+w^2)=f(u, v, w)$,

since $$a+b+c=0.$$

Hence, by (1), the plane cuts the cone in perpendicular generators. Thus any plane through the origin which is normal to a generator of the cone cuts the cone in perpendicular lines, or there are two generators of the cone at right angles to one another, and at right angles to any given generator.

Ex. 1. If a right circular cone has three mutually perpendicular generators, the semi-vertical angle is $\tan^{-1}\sqrt{2}$. (Cf. Ex. 3, § 59.)

Ex. 2. Shew that the cone whose vertex is at the origin and which passes through the curve of intersection of the sphere $x^2+y^2+z^2=3a^2$, and any plane at a distance a from the origin, has three mutually perpendicular generators.

Ex. 3. Prove that the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ has three mutually perpendicular tangent planes if

$$bc+ca+ab=f^2+g^2+h^2.$$

Ex. 4. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations to the other two.

Ans. $x=y=-z, \quad 4x=-5y=20z.$

Ex. 5. Prove that the plane $lx+my+nz=0$ cuts the cone

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$$

perpendicular lines if

$$(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0.$$

63.[1] **Equation to cone with given conic for base.** *To find the equation to the cone whose vertex is the point* (α, β, γ) *and base the conic*

$$f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0, \quad z=0.$$

---

[1] See Appendix, p. ii.

The equations to any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=z-\gamma,$$

and the line meets the plane $z=0$ in the point

$$(\alpha-l\gamma,\ \beta-m\gamma,\ 0).$$

This point is on the given conic if $f(\alpha-l\gamma, \beta-m\gamma)=0$,

i.e. if $$f(\alpha, \beta)-\gamma\left(l\frac{\partial f}{\partial\alpha}+m\frac{\partial f}{\partial\beta}\right)+\gamma^2\phi(l, m)=0, \quad \ldots\ldots\ldots\textbf{(1)}$$

where $\phi(x, y)\equiv ax^2+2hxy+by^2$. If we eliminate l and m between the equations to the line and (1), we obtain the equation to the locus of lines which pass through (α, β, γ) and intersect the conic, *i.e.* the equation to the cone. The result is

$$f(\alpha, \beta)-\gamma\left(\frac{x-\alpha}{z-\gamma}\frac{\partial f}{\partial\alpha}+\frac{y-\beta}{z-\gamma}\frac{\partial f}{\partial\beta}\right)+\gamma^2\phi\left(\frac{x-\alpha}{z-\gamma},\ \frac{y-\beta}{z-\gamma}\right)=0$$

i.e. $$(z-\gamma)^2f(\alpha, \beta)-\gamma(z-\gamma)\left(\overline{x-\alpha}\frac{\partial f}{\partial\alpha}+\overline{y-\beta}\frac{\partial f}{\partial\beta}\right)$$
$$+\gamma^2\phi(x-\alpha,\ y-\beta)=0.$$

This equation may be transformed as follows:
The coefficient of γ^2 is

$$f(\alpha, \beta)+(x-\alpha)\frac{\partial f}{\partial\alpha}+(y-\beta)\frac{\partial f}{\partial\beta}+\phi(x-\alpha,\ y-\beta)$$
$$=f(\alpha+\overline{x-\alpha},\ \beta+\overline{y-\beta})=f(x, y);$$

and the coefficient of $-z\gamma$ is

$$(x-\alpha)\frac{\partial f}{\partial\alpha}+(y-\beta)\frac{\partial f}{\partial\beta}+2f(\alpha, \beta).$$

If $f(x, y)$ be made homogeneous by means of an auxiliary variable t which is equated to unity after differentiation, we have, by Euler's theorem,

$$\alpha\frac{\partial f}{\partial\alpha}+\beta\frac{\partial f}{\partial\beta}+t\frac{\partial f}{\partial t}=2f(\alpha, \beta, t).$$

Therefore the coefficient of $-z\gamma$ becomes

$$x\frac{\partial f}{\partial\alpha}+y\frac{\partial f}{\partial\beta}+t\frac{\partial f}{\partial t}$$

Hence the equation to the cone is

$$z^2f(\alpha, \beta)-z\gamma\left(x\frac{\partial f}{\partial \alpha}+y\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}\right)+\gamma^2f(x, y)=0.$$

It is to be noted that by equating to zero the coefficient of $z\gamma$, we obtain the equation to the polar of $(\alpha, \beta, 0)$ with respect to the given conic.

(The above method is given by de Longchamps, *Problèmes de Géométrie Analytique*, vol. iii.)

Ex. 1. Find the equation to the cone whose vertex is (α, β, γ) and base (i) $ax^2+by^2=1$, $z=0$; (ii) $y^2=4ax$, $z=0$.

Ans. (i) $z^2(a\alpha^2+b\beta^2-1)-2z\gamma(a\alpha x+b\beta y-1)+\gamma^2(ax^2+by^2-1)=0$;

(ii) $z^2(\beta^2-4a\alpha)-2z\gamma\{\beta y-2a(x+\alpha)\}+\gamma^2(y^2-4ax)=0$.

Ex. 2. Find the locus of points from which three mutually perpendicular lines can be drawn to intersect the conic $z=0$, $ax^2+by^2=1$.

(If (α, β, γ) is on the locus, the cone, Ex. 1 (i), has three mutually perpendicular generators.)

Ans. $ax^2+by^2+z^2(a+b)=1$.

Ex. 3. Shew that the locus of points from which three mutually perpendicular lines can be drawn to intersect a given circle is a surface of revolution.

Ex. 4. A cone has as base the circle $z=0$, $x^2+y^2+2ax+2by=0$, and passes through the fixed point $(0, 0, c)$ on the z-axis. If the section of the cone by the plane ZOX is a rectangular hyperbola, prove that the vertex lies on a fixed circle.

Ex. 5. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, is the sphere $x^2+y^2+z^2=a^2+b^2$.

*Examples III.

1. Shew that the bisectors of the angles between the lines in which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ lie on the cone

$$\frac{u(b-c)}{x}+\frac{v(c-a)}{y}+\frac{w(a-b)}{z}=0.$$

[Five concurrent lines are necessary to determine a cone of the second degree, and the form of the given result shews that the required cone is to pass through the coordinate axes and the two bisectors. Assume, therefore, that the required equation is

$$fyz+gzx+hxy=0. \quad\text{.............................}(1)$$

The given cone is $$ax^2+by^2+cz^2=0. \quad\text{.............................}(2)$$

The necessary and sufficient conditions that the cone (1) should contain the bisectors may be stated, (i) the plane $ux+vy+wz=0$

must cut the cone (1) in perpendicular lines; (ii) the lines of section of the plane and the cones (1) and (2) must be harmonically conjugate. From (i),

$$fvw+gwu+huv=0. \qquad \text{(3)}$$

Again, four lines are harmonically conjugate if their projections on any plane are harmonically conjugate, and the equations to the projections on $z=0$ are obtained by eliminating z between the equations to the plane and cones, and hence they are

$$x^2(aw^2+cu^2)+2cuvxy+y^2(bw^2+cv^2)=0, \quad z=0;$$
$$gux^2+xy(fu+gv-wh)+fvy^2=0, \quad z=0.$$

Therefore the condition (ii) gives

$$fv(aw^2+cu^2)+gu(bw^2+cv^2)=cuv(fu+gv-wh);$$

(cf. Smith, *Conic Sections*, p. 55.)

$$\text{i.e.} \quad afvw+bgwu+chuv=0. \qquad \text{(4)}$$

From (3) and (4), we obtain

$$\frac{f}{u(b-c)}=\frac{g}{v(c-a)}=\frac{h}{w(a-b)}.\Big]$$

2. Shew that the bisectors in Ex. 1 also lie on the cone

$$\Sigma u^2x^2\{-(b-c)u^2+(c-a)v^2+(a-b)w^2\}=0.$$

3. Two cones pass through the curves $y=0$, $z^2=4ax$; $x=0$, $z^2=4by$, and they have a common vertex; the plane $z=0$ meets them in two conics that intersect in four concyclic points. Shew that the vertex lies on the surface $z^2(x/a+y/b)=4(x^2+y^2)$.

4. Planes through **OX** and **OY** include an angle α. Shew that their line of intersection lies on the cone $z^2(x^2+y^2+z^2)=x^2y^2\tan^2\alpha$.

5. Any plane whose normal lies on the cone

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0$$

cuts the surface $ax^2+by^2+cz^2=1$ in a rectangular hyperbola.

6. Find the angle between the lines given by

$$x+y+z=0, \quad \frac{yz}{b-c}+\frac{zx}{c-a}+\frac{xy}{a-b}=0.$$

7. Shew that the angle between the lines given by

$$x+y+z=0, \quad ayz+bzx+cxy=0$$

is $\pi/2$ if $a+b+c=0$, but $\pi/3$ if $1/a+1/b+1/c=0$.

8. Shew that the plane $ax+by+cz=0$ cuts the cone

$$yz+zx+xy=0$$

in two lines inclined at an angle

$$\tan^{-1}\left[\frac{\{(a^2+b^2+c^2)(a^2+b^2+c^2-2bc-2ca-2ab)\}^{\frac{1}{2}}}{bc+ca+ab}\right],$$

and by considering the value of this expression when $a+b+c=0$, shew that the cone is of revolution, and that its axis is $x=y=z$ and vertical angle $\tan^{-1}2\sqrt{2}$.

9. The axes being rectangular, prove that the cone

$$x^2=2(y^2+z^2)$$

contains an infinite number of sets of three generators mutually inclined at an angle $\pi/3$.

10. Through a fixed point **O** a line is drawn to meet three fixed intersecting planes in **P**, **Q**, **R**. If **PQ** : **PR** is constant, prove that the locus of the line is a cone whose vertex is **O**.

11. The vertex of a cone is (a, b, c) and the yz-plane cuts it in the curve $\mathsf{F}(y, z)=0$, $x=0$. Shew that the zx-plane cuts it in the curve $y=0$, $\mathsf{F}\left(\dfrac{bx}{x-a}, \dfrac{cx-az}{x-a}\right)=0$.

12. **OP** and **OQ** are two straight lines that remain at right angles and move so that the plane **OPQ** always passes through the z-axis. If **OP** describes the cone $\mathsf{F}(y/x, z/x)=0$, prove that **OQ** describes the cone

$$\mathsf{F}\left\{\frac{y}{x}, \left(-\frac{x}{z}-\frac{y^2}{zx}\right)\right\}=0.$$

13. Prove that $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $u^2/a+v^2/b+w^2/c=d$.

14. Prove that if

$$\mathsf{F}(xyz)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone, the coordinates of the vertex satisfy the equations $\mathsf{F}_x=0$, $\mathsf{F}_y=0$, $\mathsf{F}_z=0$, $\mathsf{F}_t=0$, where t is used to make $\mathsf{F}(x, y, z)$ homogeneous and is equated to unity after differentiation.

15. Prove that the equations

$$2y^2-8yz-4zx-8xy+6x-4y-2z+5=0,$$

$$2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0,$$

represent cones whose vertices are $(-7/6, 1/3, 5/6)$, $(2, 2, 1)$.

16. Find the conditions that the lines of section of the plane $lx+my+nz=0$ and the cones $fyz+gzx+hxy=0$, $ax^2+by^2+cz^2=0$, should be coincident.

$$\left(\frac{bn^2+cm^2}{fmn}=\frac{cl^2+an^2}{gnl}=\frac{am^2+bl^2}{hlm}.\right)$$

17. Find the equations to the planes through the z-axis and the lines of section of the plane $ux+vy+wz=0$ and cone $f(x, y, z)=0$, and prove that the plane touches the cone if $\mathsf{P}=0$. (The axes may be oblique.)

18. Prove that the equation to the cone through the coordinate axes and the lines of section of the cone $11x^2-5y^2+z^2=0$ and the plane $7x-5y+z=0$ is $14yz-30zx+3xy=0$, and that the other common generators of the two cones lie in the plane $11x+7y+7z=0$.

19. Prove that the common generators of the cones

$$(b^2c^2-a^4)x^2+(c^2a^2-b^4)y^2+(a^2b^2-c^4)z^2=0,$$

$$\frac{bc-a^2}{ax}+\frac{ca-b^2}{by}+\frac{ab-c^2}{cz}=0,$$

lie in the planes

$$(bc\pm a^2)x+(ca\pm b^2)y+(ab\pm c^2)z=0.$$

20. Prove that the equation to the cone through the coordinate axes and the lines in which the plane $lx+my+nz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is

$$l(bn^2+cm^2-2fmn)yz+m(cl^2+an^2-2gnl)zx+n(am^2+bl^2-2hlm)xy=0.$$

21. Prove that the equation $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ represents a cone that touches the coordinate planes, and that the equation to the reciprocal cone is $fyz+gzx+hxy=0$.

22. Prove that the equation to the planes through the origin perpendicular to the lines of section of the plane $lx+my+nz=0$ and the cone $ax^2+by^2+cz^2=0$ is

$$x^2(bn^2+cm^2)+y^2(cl^2+an^2)+z^2(am^2+bl^2)-2amnyz-2bnlzx-2clmxy=0$$

23. If a line OP, drawn through the vertex O of the cone

$$ax^2+by^2+cz^2=0,$$

is such that the two planes through OP, each of which cuts the cone in a pair of perpendicular lines, are at right angles, prove that the locus of OP is the cone

$$(2a+b+c)x^2+(2b+c+a)y^2+(2c+a+b)z^2=0.$$

CHAPTER VII.

THE CENTRAL CONICOIDS.

64. The locus of the equation

$$(1)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$(2)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

$$(3)\qquad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

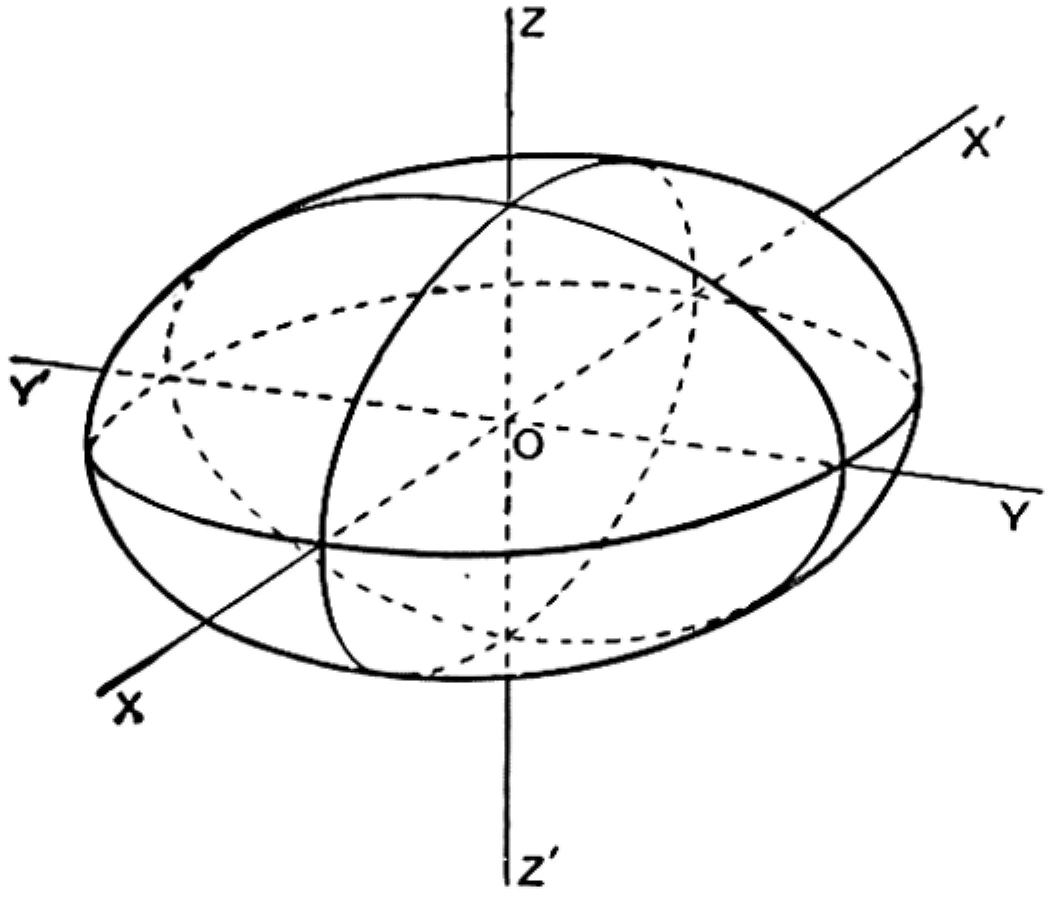

FIG. 29

We have shewn in § 9 that the equation (1) represents the surface generated by the variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\quad z=k,$$

whose centre moves along **Z′OZ**, and passes in turn through every point between $(0, 0, -c)$ and $(0, 0, +c)$. The surface is the **ellipsoid**, and is represented in fig. 29. The section by any plane parallel to a coordinate plane is an ellipse.

Similarly, we might shew that the surface represented by equation (2) is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \quad z=k,$$

whose centre moves on **Z′OZ**, passing in turn through every point on it. The surface is the **hyperboloid of one sheet**, and is represented in fig. 30. The section by any plane parallel to one of the coordinate planes **YOZ** or **ZOX** is a hyperbola.

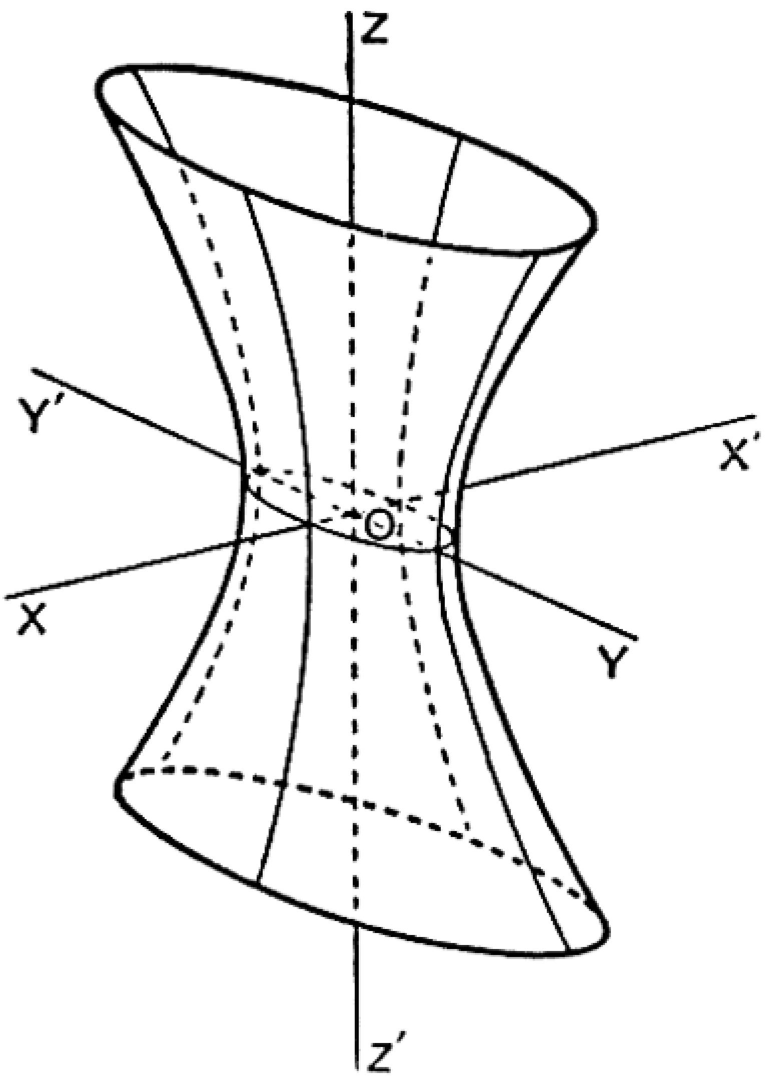

FIG. 30.

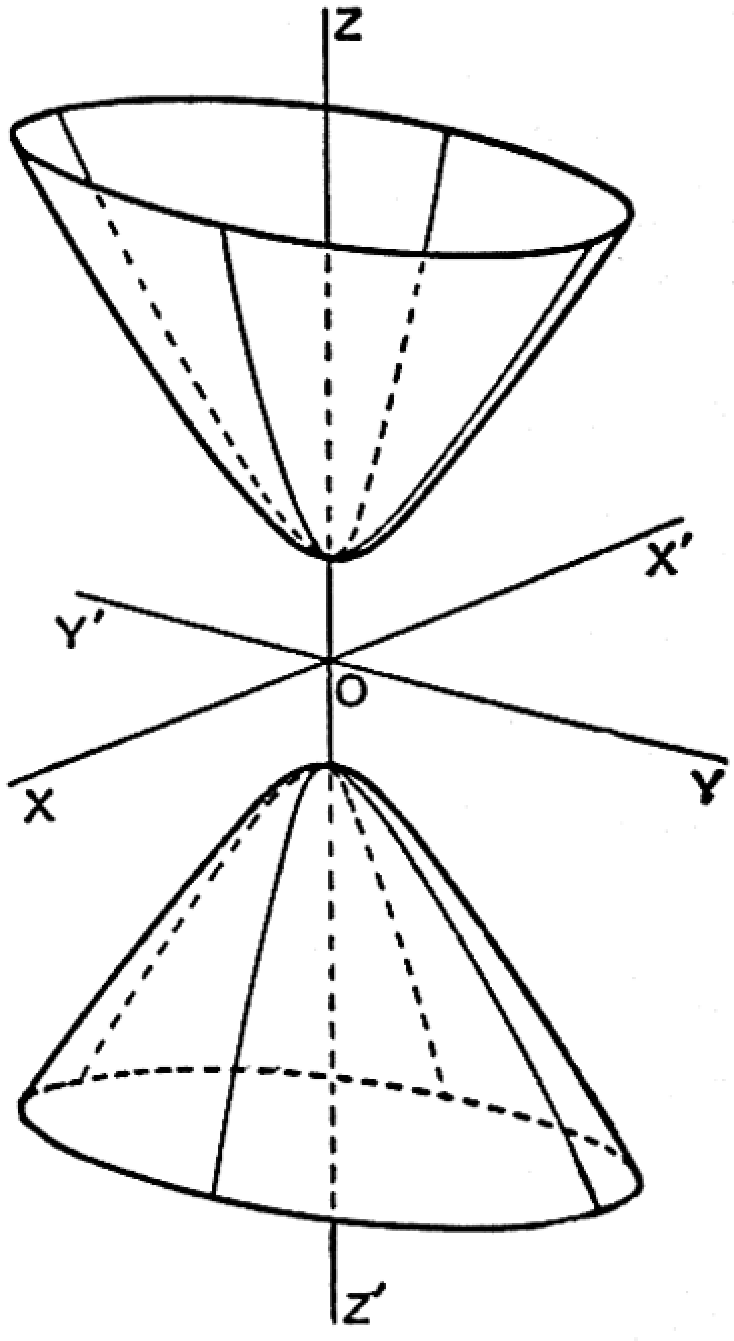

FIG. 31.

The surface given by equation (3) is also generated by a variable ellipse whose centre moves on **Z′OZ**. The ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}-1, \quad z=k.$$

and is imaginary if $-c < k < c$; hence no part of the surface lies between the planes $z = \pm c$.

The surface is the **hyperboloid of two sheets**, and is represented in fig. 31. The section by any plane parallel to one of the coordinate planes YOZ, ZOX is a hyperbola.

If (x', y', z') is any point on one of these surfaces, $(-x', -y', -z')$ is also on it; hence the origin bisects all chords of the surface which pass through it. The origin is the only point which possesses this property, and is called **the centre.** The surfaces are called **the central conicoids.**

65. Diametral planes and conjugate diameters.

An equation of the form

$$ax^2 + by^2 + cz^2 = 1$$

represents a central conicoid. The equations to any line parallel to OX are $y = \lambda$, $z = \mu$, and it meets the surface in the points

$$\left(\pm\sqrt{\frac{1 - b\lambda^2 - c\mu^2}{a}},\ \lambda,\ \mu\right),$$

and hence the plane YOZ bisects all chords parallel to OX. Any chord of the conicoid which passes through the centre is a **diameter**, and the plane which bisects a system of parallel chords is a **diametral plane**. Thus YOZ is the diametral plane which bisects chords parallel to OX, or shortly, is the diametral plane of OX. Similarly, the diametral planes ZOX, XOY bisect chords parallel to OY and OZ respectively. The three diametral planes YOZ, ZOX, XOY are such that each bisects chords parallel to the line of intersection of the other two. They are called **conjugate diametral planes**. The diameters X′OX, Y′OY, Z′OZ are such that the plane through any two bisects chords parallel to the third. They are called **conjugate diameters.**

If the axes are rectangular, the diametral planes YOZ, ZOX, XOY are at right angles to the chords which they bisect. Diametral planes which are at right angles to the chords which they bisect are **principal planes**. The lines of intersection of principal planes are **principal axes.** Hence if the axes are rectangular the equation $ax^2 + by^2 + cz^2 = 1$

represents a central conicoid referred to its principal axes as coordinate axes.

66. *A line through a given point* A, (α, β, γ) *meets a central conicoid* $ax^2+by^2+cz^2=1$ *in* P *and* Q; *to find the lengths of* AP *and* AQ.

If l, m, n are the direction-ratios of a line through A, the coordinates of the point on it whose distance from A is r are $\alpha+lr$, $\beta+mr$, $\gamma+nr$. If this point is on the conicoid,

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+a\alpha^2+b\beta^2+c\gamma^2-1=0. \quad \text{........(1)}$$

This equation gives two values of r which are the measures of AP and AQ.

Ex. 1. If OD is the diameter parallel to APQ, AP.AQ : OD² is constant.

Ex. 2. If DOD′ is any diameter of the conicoid and OR and OR′ are the diameters parallel to AD and AD′, $\frac{AD^2}{OR^2}+\frac{AD'^2}{OR'^2}$ is constant.

Ex. 3. If AD, AD′ meet the conicoid again in E and E′, $\frac{AD}{AE}+\frac{AD'}{AE'}$ is constant.

67. Tangents and tangent planes. If $a\alpha^2+b\beta^2+c\gamma^2=1$, the point A, (α, β, γ) is on the conicoid; one of the values of r given by the equation (1) of § 66 is zero, and A coincides with one of the points P or Q, say P.

If, also,
$$a\alpha l+b\beta m+c\gamma n=0,$$
the two values of r given by the equation are zero, *i.e.* P and Q coincide at the point (α, β, γ) on the surface, and the line APQ is a tangent to the conicoid at A. Hence, if A, (α, β, γ) is a point on the surface, the condition that the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{....................(2)}$$

should be a tangent at A, is

$$a\alpha l+b\beta m+c\gamma n=0. \quad \text{....................(3)}$$

If we eliminate l, m, n between (2) and (3), we obtain the equation to the locus of all the tangent lines through (α, β, γ), viz.,

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0,$$

or
$$a\alpha x+b\beta y+c\gamma z=1.$$

Hence the tangent lines at $(\alpha. \beta, \gamma)$ lie in the plane

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is the **tangent plane** at (α, β, γ).

68. *To find the condition that the plane* $lx+my+nz=p$ *should touch the conicoid* $ax^2+by^2+cz^2=1$.

If the point of contact is (α, β, γ), the given plane is represented by the two equations

$$a\alpha x+b\beta y+c\gamma z=1,$$
$$lx+my+nz=p.$$

Therefore $\alpha=\dfrac{l}{ap},\quad \beta=\dfrac{m}{bp},\quad \gamma=\dfrac{n}{cp};$

and, since (α, β, γ) is on the conicoid,

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2.$$

Cor. The two tangent planes which are parallel to $lx+my+nz=0$ are given by

$$lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

Ex. 1. Find the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid.

If the axes are rectangular and

$$l_r x+m_r y+n_r z=\sqrt{\frac{l_r^2}{a}+\frac{m_r^2}{b}+\frac{n_r^2}{c}},\quad r=1, 2, 3,$$

represent three mutually perpendicular tangent planes, squaring and adding, we obtain

$$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Hence the common point of the planes lies on a sphere concentric with the conicoid. (It is called *the director sphere*.)

Ex. 2. Prove that the equation to the two tangent planes to the conicoid $ax^2+by^2+cz^2=1$ which pass through the line

$$u\equiv lx+my+nz-p=0,\quad u'\equiv l'x+m'y+n'z-p'=0,\ \text{is}$$

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0.$$

(Use the condition that $u+\lambda u'=0$ should be a tangent plane.)

Ex. 3. Find the equations to the tangent planes to

$$2x^2-6y^2+3z^2=5$$

which pass through the line $x+9y-3z=0=3x-3y+6z-5$.

Ans. $2x-12y+9z=5, \quad 4x+6y+3z=5$.

Ex. 4. If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation, referred to rectangular axes, is

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

passes through the fixed point $(0, 0, k)$, shew that it lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

Ex. 5. Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone $(\alpha x+\beta y+\gamma z)^2=\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}$.

Prove that the reciprocal of this cone is the cone

$$(ax^2+by^2+cz^2)(a\alpha^2+b\beta^2+c\gamma^2-1)-(a\alpha x+b\beta y+c\gamma z)^2=0,$$

and hence shew that the tangent planes envelope the cone

$$(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)-(a\alpha x+b\beta y+c\gamma z-1)^2=0.$$

69. The polar plane. We now proceed to define the polar of a point with respect to a conicoid, and to find its equation.

Definition. If any secant, APQ, through a given point A, meets a conicoid in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q, is the polar of A with respect to the conicoid.

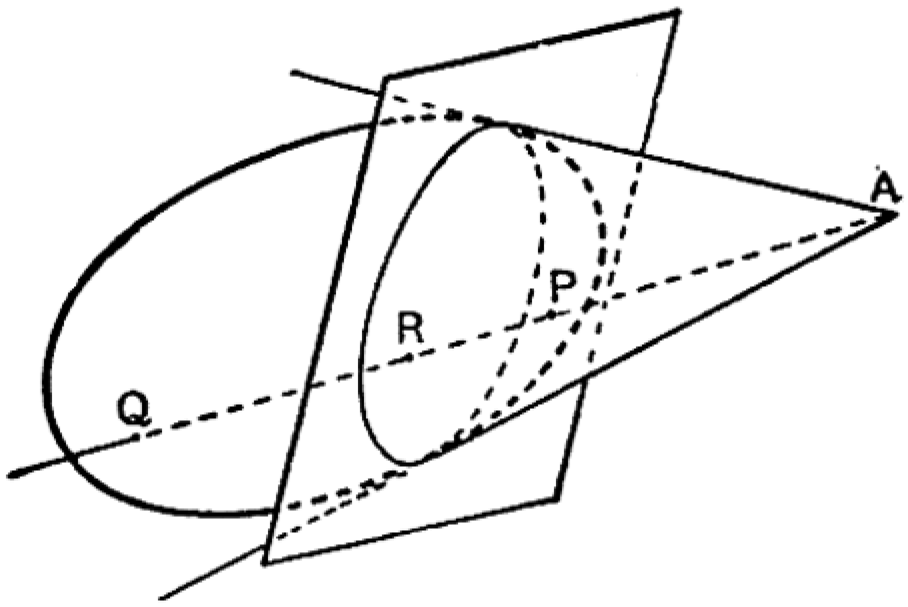

FIG. 32

Let A, R (fig. 32) be the points (α, β, γ), (ξ, η, ζ), and let APQ have direction-ratios l, m, n. Then the equations to APQ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

and, as in § 66, r_1, r_2, the measures of AP and AQ, are the roots of

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

Let ρ be the measure of AR. Then, since AP, AR, AQ are in harmonic progression,

$$\rho=\frac{2r_1r_2}{r_1+r_2}=-\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{a\alpha l+b\beta m+c\gamma n}.$$

And from the equations to the line

$$\xi-\alpha=l\rho,\quad \eta-\beta=m\rho,\quad \zeta-\gamma=n\rho,$$

therefore

$$(\xi-\alpha)a\alpha+(\eta-\beta)b\beta+(\zeta-\gamma)c\gamma=-(a\alpha^2+b\beta^2+c\gamma^2-1).$$

Hence the locus of (ξ, η, ζ) is the plane given by

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is called the **polar plane** of (α, β, γ).

Cor. If A is on the surface, the polar plane of A is the tangent plane at A

The student cannot have failed to notice the similarity between the equations to corresponding loci in the plane and in space. There is a close analogy between the equations to the line and the plane, the circle and the sphere, the ellipse and the ellipsoid, the tangent or polar and the tangent plane or polar plane. Examples of this analogy will constantly recur, and it is well to note these and make use of the analogy as an aid to remember useful results.

70. Polar lines. It is evident that if the polar plane of (α, β, γ) passes through (ξ, η, ζ), then the polar plane of (ξ, η, ζ) passes through (α, β, γ). Hence if the polar plane of any point on a line AB passes through a line PQ, then the polar plane of any point on PQ passes through that point on AB, and therefore passes through AB. The lines AB and PQ are then said to be **polar lines** with respect to the conicoid.

The polar plane of $(\alpha+lr, \beta+mr, \gamma+nr)$, any point on the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is $$a\alpha x+b\beta y+c\gamma z-1+r(alx+bmy+cnz)=0,$$

and, evidently, for all values of r, passes through the line

$$a\alpha x + b\beta y + c\gamma z - 1 = 0 = alx + bmy + cnz.$$

This is therefore the polar of the line (1).

Ex. 1. If P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2) are any points, the polar of PQ with respect to $ax^2+by^2+cz^2=1$ is given by

$$axx_1+byy_1+czz_1=1, \quad axx_2+byy_2+czz_2=1.$$

(Hence if P and Q are on the conicoid the polar of PQ is the line of intersection of the tangent planes at P and Q.)

Ex. 2. Prove that the polar of a given line is the chord of contact of the two tangent planes through the line.

Ex. 3. Find the equations to the polar of the line

$$-2x=25y-1=2z$$

with respect to the conicoid $2x^2-25y^2+2z^2=1$. Prove that it meets the conicoid in two real points P and Q, and verify that the tangent planes at P and Q pass through the given line.

Ans. $\dfrac{x}{1}=\dfrac{y+1}{0}=\dfrac{z+1}{1}$.

Ex. 4. Find the locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to $ax^2+by^2+cz^2=1$; (rectangular axes).

Ans. $\Sigma \dfrac{\alpha}{x-\alpha}\left(\dfrac{1}{b}-\dfrac{1}{c}\right)=0$.

Ex. 5. Prove that lines through (α, β, γ) at right angles to their polars with respect to $\dfrac{x^2}{a+b}+\dfrac{y^2}{2a}+\dfrac{z^2}{2b}=1$ generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

What is the peculiarity of the case when $a=b$?

Ex. 6. Find the conditions that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'},$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

Ans. $\Sigma a\alpha\alpha'=1$, $\Sigma a\alpha' l=0$, $\Sigma a\alpha l'=0$, $\Sigma all'=0$.

Ex. 7. Find the condition that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ should intersect the polar of the line $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$ with respect to the conicoid $ax^2+by^2+cz^2=1$.

Ans. $(a\alpha l'+b\beta m'+c\gamma n')(a\alpha' l+b\beta' m+c\gamma' n)$
$=(all'+bmm'+cnn')(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1)$.

Ex. 8. Prove that if AB intersects the polar of PQ, then PQ intersects the polar of AB. (AB and PQ are then said to be **conjugate** with respect to the conicoid.)

71. Section with a given centre. If (α, β, γ) is the mid-point of the chord whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \text{.......................(1)}$$

the equation (1) of § 66 is of the form $r^2=k^2$, and therefore

$$a\alpha l+b\beta m+c\gamma n=0. \quad \text{.......................(2)}$$

Hence all chords which are bisected at (α, β, γ) lie in the plane

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

This plane meets the surface in a conic of which (α, β, γ) is the centre.

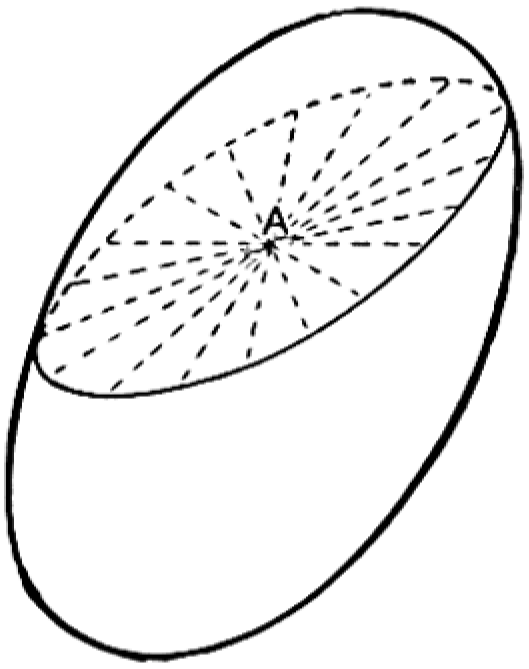

FIG. 83.

Compare the equation to the chord of the conic $ax^2+by^2=1$ whose mid-point is (α, β).

Ex. 1. Find the equation to the plane which cuts $x^2+4y^2-5z^2=1$ in a conic whose centre is at the point (2, 3, 4).

Ans. $x+6y-10z+20=0$.

Ex. 2. The locus of the centres of parallel plane sections of a conicoid is a diameter.

Ex. 3. The line joining a point P to the centre of a conicoid passes through the centre of the section of the conicoid by the polar plane of P.

Ex. 4. The centres of sections of a central conicoid that are parallel to a given line lie on a fixed plane.

Ex. 5. The centres of sections that pass through a given line lie on a conic.

Ex. 6. The centres of sections that pass through a given point lie on a conicoid.

Ex. 7. Find the locus of centres of sections of $ax^2+by^2+cz^2=1$ which touch $\alpha x^2+\beta y^2+\gamma z^2=1$.

Ans. $(ax^2+by^2+cz^2)^2=\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}$.

72. Locus of mid-points of a system of parallel chords. It follows from equations (1) and (2) of § 71 that the mid-points of chords which are parallel to a fixed line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

lie in the plane $alx+bmy+cnz=0$.

This is therefore the diametral plane which bisects the parallel chords (fig. 34).

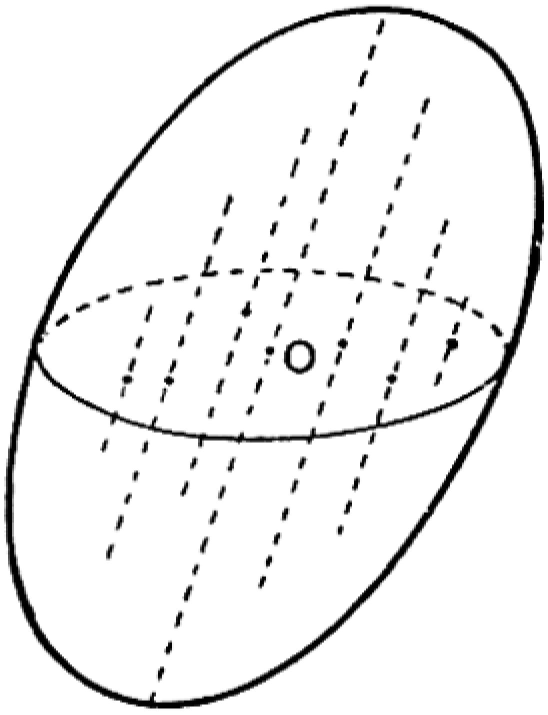

FIG. 34.

Compare the equation to the locus of the mid-points of parallel chords of the ellipse $ax^2+by^2=1$.

Ex. 1. Find the locus of the mid-points of chords of the conicoid $ax^2+by^2+cz^2=1$ which pass through the point (f, g, h).

Ans. $ax(x-f)+by(y-g)+cz(z-h)=0$.

Ex. 2. Prove that the mid-points of chords of $ax^2+by^2+cz^2=1$ which are parallel to $x=0$ and touch $x^2+y^2+z^2=r^2$ lie on the surface

$$by^2(bx^2+by^2+cz^2-br^2)+cz^2(ax^2+by^2+cz^2-cr^2)=0.$$

73. The locus of the tangents drawn from a given point. When the secant APQ, (fig. 32), becomes a tangent, P, Q, R coincide at the point of contact, and hence the points of contact of all the tangents from A lie on the polar plane of A, and therefore on the conic in which that plane cuts the surface. The locus of the tangents from A is therefore the cone generated by lines which pass through A and intersect the conic in which the polar plane of A cuts the conicoid. This cone is the **enveloping cone** whose vertex is A. We may find its equation as follows: If A is (α, β, γ), and the line APQ, whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

meets the surface in coincident points, the equation **(1)** of § 36 has equal roots, and therefore

$$(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$
$$=(a\alpha l+b\beta m+c\gamma n)^2. \quad \text{.........(1)}$$

The locus of APQ is therefore the cone whose equation is

$$[a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2][a\alpha^2+b\beta^2+c\gamma^2-1]$$
$$=[a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)]^2.$$

If $\quad \text{S}\equiv ax^2+by^2+cz^2-1, \quad \text{S}_1\equiv a\alpha^2+b\beta^2+c\gamma^2-1,$

and $\quad \text{P}\equiv a\alpha x+b\beta y+c\gamma z-1,$

this equation may be written

$$(\text{S}-2\text{P}+\text{S}_1)\text{S}_1=(\text{P}-\text{S}_1)^2, \ i.e.\ \text{SS}_1=\text{P}^2, \text{ or}$$
$$(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)=(a\alpha x+b\beta y+c\gamma z-1)^2.$$

Compare the equation to the pair of tangents from the point (α, β) to the ellipse $ax^2+by^2=1$.

Ex. 1. Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the surface $ax^2+by^2+cz^2=1$.

Ans. $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$.

Ex. 2. Lines drawn from the centre of a central conicoid parallel to the generators of the enveloping cone whose vertex is A generate a cone which intersects the conicoid in two conics whose planes are parallel to the polar plane of A.

Ex. 3. Through a fixed point $(k, 0, 0)$ pairs of perpendicular tangent lines are drawn to the surface $ax^2+by^2+cz^2=1$. Shew that the plane through any pair touches the cone

$$\frac{(x-k)^2}{(ak^2-1)(b+c)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Ex. 4. The plane $z=a$ meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$ in a conic which has a focus at the point $(0, 0, a)$.

Ex. 5. Find the locus of a luminous point if the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ casts a circular shadow on the plane $z=0$.

Ans. $x=0, \ \dfrac{y^2}{b^2-a^2}+\dfrac{z^2}{c^2}=1; \quad y=0, \ \dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2}=1.$

Ex. 6. If $S=0$, $u=0$, $v=0$ are the equations to a conicoid and two planes, prove that $S+\lambda uv=0$ represents a conicoid which passes through the conics in which the given planes cut the given conicoid, and interpret the equation $S+\lambda u^2=0$.

Ex. 7. Prove that if a straight line has three points on a conicoid, it lies wholly on the conicoid.

(The equation (1), § 66, is an identity.)

Ex. 8. A conicoid passes through a given point A and touches a given conicoid S at all points of the conic in which it is met by the polar plane of A. Prove that all the tangents from A to S lie on it. Hence find the equation to the enveloping cone of S whose vertex is A.

Ex. 9. The section of the enveloping cone of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose vertex is P by the plane $z=0$ is (i) a parabola, (ii) a rectangular hyperbola. Find the locus of P.

Ans. (i) $z=\pm c$, (ii) $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1$.

74. The locus of the tangents which are parallel to a given line. Suppose that PQ is any chord and that M is its mid-point. Then if the line PQ moves parallel to itself till it meets the surface in coincident points, it becomes a tangent and M coincides with the point of contact. Therefore the point of contact of a tangent which is parallel to a given line lies on the diametral plane which bisects all chords parallel to the line. This plane cuts the surface in a conic, and the locus of the tangents parallel to the given line is therefore the cylinder generated by the parallels to the given line which pass through the conic.

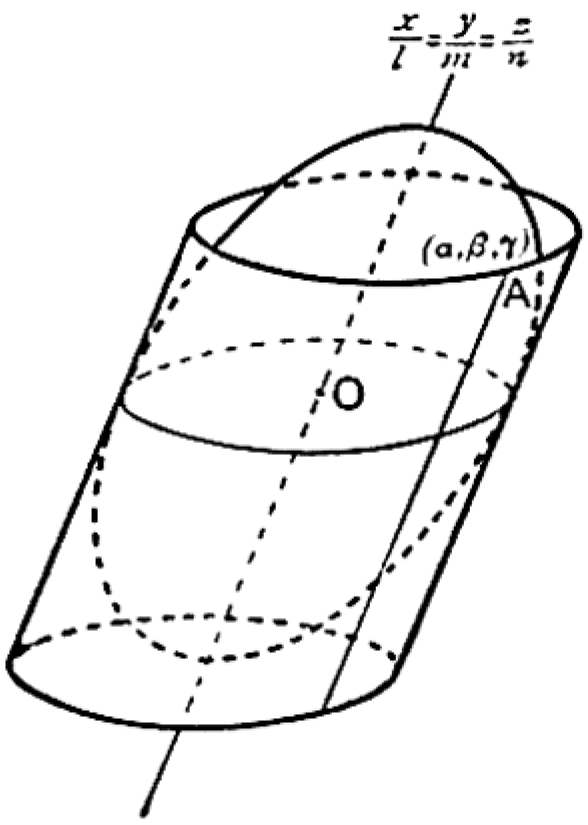

FIG. 35.

Let (α, β, γ) (fig. 35) be any point on a tangent parallel to a given line $x/l = y/m = z/n$.

Then since, by § 73 (1), the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

touches the surface if

$$(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1) = (a\alpha l+b\beta m+c\gamma n)^2,$$

the locus of (α, β, γ) is given by

$$(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1) = (alx+bmy+cnz)^2.$$

This equation therefore represents the **enveloping cylinder** which is the locus of the tangents.

The enveloping cylinder may be considered to be a limiting case of the enveloping cone whose vertex is the point P, (lr, mr, nr) on the line $x/l = y/m = z/n$, as r tends to infinity. By § 73, the equation to the cone is

$$(ax^2+by^2+cz^2-1)\left(al^2+bm^2+cn^2-\frac{1}{r^2}\right) = \left(alx+bmy+cnz-\frac{1}{r}\right)^2,$$

whence the equation to the cylinder can be at once deduced.

Ex. 1. Prove that the enveloping cylinders of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, whose generators are parallel to the lines

$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c},$$

meet the plane $z=0$ in circles.

Ex. 2. Prove that the polar of a line AB is the line of intersection of the planes of contact of the enveloping cone whose vertex is A and the enveloping cylinder whose generators are parallel to AB.

75. Normals.

In discussing the properties of the normals we shall confine our attention to the normals to the ellipsoid, the most familiar of the central conicoids.

Consider the ellipsoid whose equation, referred to rectangular axes, is $x^2/a^2+y^2/b^2+z^2/c^2=1$. If the plane

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma, \quad (p>0),$$

is a tangent plane whose point of contact is (x', y', z'), we have, as in § 68,

$$\cos\alpha = \frac{px'}{a^2}, \quad \cos\beta = \frac{py'}{b^2}, \quad \cos\gamma = \frac{pz'}{c^2};$$

that is, the direction-cosines of the outward-drawn normal at (x', y', z') are $\frac{px'}{a^2}, \frac{py'}{b^2}, \frac{pz'}{c^2}$, where p is the perpendicular from the centre to the tangent plane at the point. The equations to the normal at (x', y', z') are therefore

$$\frac{x-x'}{\frac{px'}{a^2}}=\frac{y-y'}{\frac{py'}{b^2}}=\frac{z-z'}{\frac{pz'}{c^2}} \quad (=r).$$

Ex. 1. If the normal at P meets the principal planes in G_1, G_2, G_3, shew that

$$PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2.$$

Putting 0 for x in the equations to the normal, we obtain

$$r=PG_1=-\frac{a^2}{p}, \text{ etc.}$$

Ex. 2. If $PG_1{}^2+PG_2{}^2+PG_3{}^2=k^2$, find the locus of P.

Ans. The curve of intersection of the given ellipsoid and the ellipsoid $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^2}{a^4+b^4+c^4}$.

Ex. 3. Find the length of the normal chord through P, and prove that if it is equal to $4PG_3$, P lies on the cone

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0.$$

Ans. $2\Big/\left(p^3\Sigma\frac{x'^2}{a^6}\right)$.

Ex. 4. The normal at a variable point P meets the plane XOY in A, and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

Find the locus of R if OR is drawn from the centre equal and parallel to AP. *Ans.* $a^2x^2+b^2y^2+c^2z^2=c^4$.

Ex. 5. If the normals at P and Q, points on the ellipsoid, intersect. PQ is at right angles to its polar with respect to the ellipsoid.

76. The normals from a given point. If the normal at (x', y', z') passes through a given point (α, β, γ), then

$$\frac{\alpha-x'}{\frac{x'}{a^2}}=\frac{\beta-y'}{\frac{y'}{b^2}}=\frac{\gamma-z'}{\frac{z'}{c^2}}, \quad \dots\dots\dots\dots\dots\dots(1)$$

and if each of these fractions is equal to λ,

$$x'=\frac{a^2\alpha}{a^2+\lambda}, \quad y'=\frac{b^2\beta}{b^2+\lambda}, \quad z'=\frac{c^2\gamma}{c^2+\lambda}. \quad \dots\dots\dots(2)$$

Therefore, since (x', y', z') is on the ellipsoid,

$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1. \quad\text{..............(3)}$$

This equation gives six values of λ, to each of which corresponds a point (x', y', z'), and therefore there are six points on the ellipsoid the normals at which pass through (α, β, γ).

Ex. 1. Prove that equation (3) gives at least two real values of λ. (If $F(\lambda)\equiv(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2-\Sigma a^2\alpha^2(\lambda+b^2)^2(\lambda+c^2)^2$, $F(\lambda)$ is negative when $\lambda=-a^2, -b^2, -c^2$, and is positive when $\lambda=\pm\infty$.)

Ex. 2. Prove that four normals to the ellipsoid pass through any point of the curve of intersection of the ellipsoid and the conicoid

$$x^2(b^2+c^2)+y^2(c^2+a^2)+z^2(a^2+b^2)=b^2c^2+c^2a^2+a^2b^2.$$

It follows from equations (1) that the feet of the normals from (α, β, γ) to the ellipsoid lie on the three cylinders

$$b^2z(\beta-y)=c^2y(\gamma-z),\quad c^2x(\gamma-z)=a^2z(\alpha-x),$$
$$a^2y(\alpha-x)=b^2x(\beta-y).$$

Compare the equation to the rectangular hyperbola through the feet of the normals from the point (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

These cylinders have a common curve of intersection, and equations (2) express the coordinates of any point on it in terms of a parameter λ. The points where the curve meets a given plane

$$ux+vy+wz+d=0$$

are given by $\quad \dfrac{ua^2\alpha}{a^2+\lambda}+\dfrac{vb^2\beta}{b^2+\lambda}+\dfrac{wc^2\gamma}{c^2+\lambda}+d=0,$

and as this determines three values of λ, the plane meets the curve in three points, and the curve is therefore a *cubic curve.* The feet of the normals from (α, β, γ) to the ellipsoid are therefore the six points of intersection of the ellipsoid and a certain cubic curve.

If the normal at (x', y', z') passes through (α, β, γ) and has direction-cosines l, m, n,

$$l=\frac{px'}{a^2}=\frac{p\alpha}{a^2+\lambda},\quad m=\frac{p\beta}{b^2+\lambda},\quad n=\frac{p\gamma}{c^2+\lambda},$$

and therefore

$$\frac{\alpha}{l}(b^2-c^2)+\frac{\beta}{m}(c^2-a^2)+\frac{\gamma}{n}(a^2-b^2)=0.$$

This shews that the normal whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is a generator of the cone

$$\frac{\alpha(b^2-c^2)}{x-\alpha}+\frac{\beta(c^2-a^2)}{y-\beta}+\frac{\gamma(a^2-b^2)}{z-\gamma}=0.$$

Hence the six normals from (α, β, γ) lie on a cone of the second degree.

Ex. 3. If P is the point (α, β, γ), prove that the line PO, the parallels through P to the axes, and the perpendicular from P to its polar plane, lie on the cone.

Ex. 4. Shew that the cubic curve lies on the cone.

Ex. 5. Prove that the feet of the six normals from (α, β, γ) lie on the curve of intersection of the ellipsoid and the cone

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0.$$

Ex. 6. The generators of the cone which contains the normals from a given point to an ellipsoid are at right angles to their polars with respect to the ellipsoid.

Ex. 7. A is a fixed point and P a variable point such that its polar plane is at right angles to AP. Shew that the locus of P is the cubic curve through the feet of the normals from A.

Ex. 8. If P, Q, R ; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid, and the plane PQR is given by $lx+my+nz=p$, then the plane P′Q′R′ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

(If P′Q′R′ is given by $l'x+m'y+n'z=p'$, equation (3) of § 76 is the same as

$$\left(\frac{a^2\alpha l}{a^2+\lambda}+\frac{b^2\beta m}{b^2+\lambda}+\frac{c^2\gamma n}{c^2+\lambda}-p\right)\left(\frac{a^2\alpha l'}{a^2+\lambda}+\frac{b^2\beta m'}{b^2+\lambda}+\frac{c^2\gamma n'}{c^2+\lambda}-p'\right)=0.)$$

Ex. 9. If A, A′ are the poles of the planes PQR, P′Q′R′,

$$AA'^2-OA^2-OA'^2=2(a^2+b^2+c^2).$$

77. Conjugate diameters and conjugate diametral planes of the ellipsoid. If the equation to the ellipsoid is $x^2/a^2+y^2/b^2+z^2/c^2=1$, the axes are conjugate diameters and the coordinate planes are conjugate diametral planes, (§ 65). If P, (x_1, y_1, z_1) is any point on the ellipsoid, the diametral plane of OP has for its equation, (§ 72),

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0.$$

Let Q, (x_2, y_2, z_2) be any point on this plane and on the ellipsoid, then

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0.$$

Hence, if Q is on the diametral plane of OP, P is on the diametral plane of OQ.

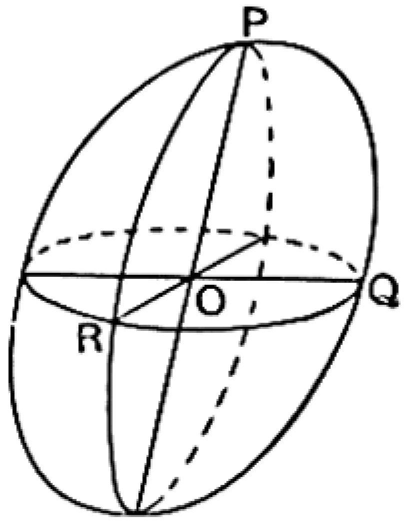

FIG. 36.

If the diametral planes of OP and OQ intersect in the diameter OR, (fig. 36), R is on the diametral planes of OP and OQ, and therefore P and Q are on the diametral plane of OR; that is, the diametral plane of OR is the plane OPQ. Thus the planes QOR, ROP, POQ are the diametral planes of OP, OQ, OR respectively, and they are therefore conjugate diametral planes, and OP, OQ, OR are conjugate diameters.

If R is (x_3, y_3, z_3), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1, \quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1, \quad \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1, \quad (\text{A}')$$

$$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0, \quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0,$$

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B}')$$

These correspond exactly to equations (A) and (B) of § 53, and shew that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \quad \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \quad \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

are the direction-cosines of three mutually perpendicular lines referred to rectangular axes. Therefore, as in § 53, we deduce,

$$\left.\begin{array}{l} x_1^2+x_2^2+x_3^2=a^2,\\ y_1^2+y_2^2+y_3^2=b^2,\\ z_1^2+z_2^2+z_3^2=c^2; \end{array}\right\}\text{......(C}'\text{)} \quad \left.\begin{array}{l} y_1z_1+y_2z_2+y_3z_3=0,\\ z_1x_1+z_2x_2+z_3x_3=0,\\ x_1y_1+x_2y_2+x_3y_3=0; \end{array}\right\}\text{......(D}'\text{)}$$

$$\frac{x_1}{a}=\pm\frac{(y_2z_3-z_2y_3)}{bc},\quad \frac{y_1}{b}=\pm\frac{(z_2x_3-x_2z_3)}{ca},\quad \frac{z_1}{c}=\pm\frac{(x_2y_3-y_2x_3)}{ab},$$

$$\frac{x_2}{a}=\pm\frac{(y_3z_1-z_3y_1)}{bc},\ \text{etc., etc.; ....(E}'\text{)}$$

$$\begin{vmatrix} x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\\ x_3, & y_3, & z_3 \end{vmatrix}=\pm abc.$$

If the axes to which the ellipsoid is referred are rectangular, equations (C′) give, on adding,

$$\text{OP}^2+\text{OQ}^2+\text{OR}^2=a^2+b^2+c^2.$$

Hence the sum of the squares on any three conjugate semi-diameters is constant. From the last equation we deduce that the volume of the parallelepiped which has OP, OQ, OR for coterminous edges is constant and equal to abc. Again, if A_1, A_2, A_3 are the areas QOR, ROP, POQ and l_r, m_r, n_r, $(r=1, 2, 3)$, are the direction-cosines of the normals to the planes QOR, ROP, POQ, projecting A_1 on the plane $x=0$, we obtain

$$l_1\text{A}_1=\frac{y_2z_3-z_2y_3}{2}=\pm\frac{bcx_1}{2a},\quad \text{by (E}'\text{)};$$

similarly, $$m_1\text{A}_1=\pm\frac{cay_1}{2b},\quad n_1\text{A}_1=\pm\frac{abz_1}{2c};$$

$$l_2\text{A}_2=\pm\frac{bcx_2}{2a},\quad m_2\text{A}_2=\pm\frac{cay_2}{2b},\quad n_2\text{A}_2=\pm\frac{abz_2}{2c};$$

$$l_3\text{A}_3=\pm\frac{bcx_3}{2a},\quad m_3\text{A}_3=\pm\frac{cay_3}{2b},\quad n_3\text{A}_3=\pm\frac{abz_3}{2c}.$$

Therefore, squaring and adding, we have, by (C′),

$$\text{A}_1^2+\text{A}_2^2+\text{A}_3^2=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2).$$

Ex. 1. Find the equation to the plane **PQR**.

If the equation is $lx+my+nz=p$, then $lx_1+my_1+nz_1=p$,

$$lx_2+my_2+nz_2=p, \quad lx_3+my_3+nz_3=p.$$

Multiply by x_1, x_2, x_3 respectively, and add; then by (C′) and (D′),

$$la^2=p(x_1+x_2+x_3), \text{ etc.}$$

The required equation is therefore

$$\frac{x(x_1+x_2+x_3)}{a^2}+\frac{y(y_1+y_2+y_3)}{b^2}+\frac{z(z_1+z_2+z_3)}{c^2}=1.$$

Ex. 2. Shew that the plane **PQR** touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$ at the centroid of the triangle **PQR**.

Ex. 3. Prove that the pole of the plane **PQR** lies on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=3.$$

Ex. 4. The locus of the foot of the perpendicular from the centre to the plane through the extremities of three conjugate semi-diameters is $a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2$.

Ex. 5. Prove that the sum of the squares of the projections of **OP**, **OQ**, **OR**, (i) on a given line, (ii) on a given plane, is constant.

Ex. 6. Shew that any two sets of conjugate diameters lie on a cone of the second degree. (Cf. § 59, Ex. 6.)

Ex. 7. Shew that any two sets of conjugate diametral planes touch a cone of the second degree. (Apply § 61, Ex. 4.)

Ex. 8. If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

If r is the length of one of the equal conjugate diameters,

$$3r^2=a^2+b^2+c^2,$$

and
$$\frac{l^2+m^2+n^2}{r^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2},$$

where l, m, n are the direction-cosines. Therefore the diameter is a generator of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2},$$

or
$$\frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}(2c^2-a^2-b^2)=0.$$

Ex. 9. Shew that the plane through a pair of equal conjugate diameters touches the cone $\Sigma\frac{x^2}{a^2(2a^2-b^2-c^2)}=0$.

Ex. 10. If λ, μ, ν are the angles between a set of equal conjugate diameters,

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=\frac{3\Sigma(b^2-c^2)^2}{2(a^2+b^2+c^2)^2}.$$

Ex. 11. If **OP**, **OQ**, **OR** are equal conjugate diameters, and **S** is the pole of the plane **PQR**, the tetrahedron **SPQR** has any pair of opposite edges at right angles.

Ex. 12. If OP, OQ, OR are conjugate diameters and p_1, p_2, p_3; π_1, π_2, π_3 are their projections on any two given lines, $p_1\pi_1+p_2\pi_2+p_3\pi_3$ is constant.

Ex. 13. If, through a given point, chords are drawn parallel to OP, OQ, OR, the sum of the squares of the ratios of the respective chords to OP, OQ, OR is constant.

Ex. 14. The locus of the point of intersection of three tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to conjugate diametral planes of $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$. What does this theorem become when $\alpha=\beta=\gamma$?

Ex. 15. Shew that conjugate diameters satisfy the condition of Ex. 8, § 70, for conjugate lines.

Since the plane POQ, (fig. 36), bisects all chords of the conicoid which are parallel to OR, the line OQ bisects all chords of the conic ROQ which are parallel to OR. Similarly OR bisects all chords of the conic which are parallel to OQ; and therefore OR and OQ are conjugate diameters of the ellipse ROQ. But Q is any point on the ellipse; therefore OP and any pair of conjugate diameters of the ellipse in which the diametral plane of OP cuts the ellipsoid are conjugate diameters of the ellipsoid.

Ex. 16. P is any point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and 2α and 2β are the principal axes of the section of the ellipsoid by the diametral plane of OP. Prove that $OP^2=a^2+b^2+c^2-\alpha^2-\beta^2$, and that $\alpha\beta p=abc$, where p is the perpendicular from O to the tangent plane at P.

Ex. 17. If 2α and 2β are the principal axes of the section of the ellipsoid by the plane $lx+my+nz=0$, prove that

$$\alpha^2+\beta^2=\frac{a^2(b^2+c^2)l^2+b^2(c^2+a^2)m^2+c^2(a^2+b^2)n^2}{a^2l^2+b^2m^2+c^2n^2},$$

$$\alpha^2\beta^2=\frac{a^2b^2c^2(l^2+m^2+n^2)}{a^2l^2+b^2m^2+c^2n^2}.$$

Ex. 18. If P, (x_1, y_1, z_1) is a point on the ellipsoid and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are extremities of the principal axes of the section of the ellipsoid by the diametral plane of OP, prove that

$$\frac{\xi_1\xi_2}{a^2(b^2-c^2)}=\frac{\eta_1\eta_2}{b^2(c^2-a^2)}=\frac{\zeta_1\zeta_2}{c^2(a^2-b^2)},$$

$$(b^2-c^2)\frac{x_1}{\xi_1}+(c^2-a^2)\frac{y_1}{\eta_1}+(a^2-b^2)\frac{z_1}{\zeta_1}=0.$$

Conjugate diameters of the hyperboloids. The equation of a hyperboloid of one sheet referred to three conjugate diameters as coordinate axes is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}=1$. Hence it appears that the x- and y-axes meet the surface in real points $(\pm\alpha, 0, 0)$, $(0, \pm\beta, 0)$, and that the z-axis does not intersect the surface. The z-axis, however, intersects the hyperboloid of two sheets whose equation is $-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ at the points $(0, 0, \pm\gamma)$, and these points are taken as the extremities of the third of the three conjugate diameters.

Hence, if $\mathsf{P}, (x_1, y_1, z_1)$, $\mathsf{Q}, (x_2, y_2, z_2)$, $\mathsf{R}, (x_3, y_3, z_3)$ are the extremities of a set of conjugate semi-diameters of the hyperboloid of one sheet,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

it follows, as for the ellipsoid, that

$$x_1^2+x_2^2-x_3^2=a^2, \quad y_1^2+y_2^2-y_3^2=b^2, \quad z_1^2+z_2^2-z_3^2=-c^2, \text{ etc.};$$

and therefore, that if the axes are rectangular,

$$\mathsf{OP}^2+\mathsf{OQ}^2-\mathsf{OR}^2=a^2+b^2-c^2$$

and $$\mathsf{A}_1^2+\mathsf{A}_2^2-\mathsf{A}_3^2=\tfrac{1}{4}(b^2c^2+c^2a^2-a^2b^2).$$

Similarly, if one of a set of three conjugate diameters of the hyperboloid of two sheets, $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, intersects the surface, the other two do not, but they intersect the hyperboloid of one sheet,

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the points of intersection are taken as their extremities. Hence if P, Q, R are the extremities, and the axes are rectangular, we have

$$\mathsf{OP}^2-\mathsf{OQ}^2-\mathsf{OR}^2=a^2-b^2-c^2$$

and $$\mathsf{A}_1^2-\mathsf{A}_2^2-\mathsf{A}_3^2=\tfrac{1}{4}(b^2c^2-c^2a^2-a^2b^2).$$

THE CONE.

78. A homogeneous equation of the form

$$ax^2+by^2+cz^2=0$$

represents a cone. If (x', y', z') is any point on the cone, $(-x', -y', -z')$ lies also on the cone, and therefore we may consider the cone as a central surface, the vertex being the centre. The coordinate planes are conjugate diametral planes and the coordinate axes are conjugate diameters.

We easily find, as in the case of the other central conicoids, the following results:

The tangent plane at (x', y', z') has for its equation

$$axx'+byy'+czz'=0.$$

The plane $lx+my+nz=0$ touches the cone if

$$l^2/a+m^2/b+n^2/c=0.$$

The polar plane of P, (α, β, γ) is given by

$$a\alpha x+b\beta y+c\gamma z=0.$$

The section whose centre is at (α, β, γ) has the equation

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

The diametral plane of the line $x/l=y/m=z/n$ is

$$alx+bmy+cnz=0.$$

The locus of the tangents drawn from P, (α, β, γ) is the pair of tangent planes whose line of intersection is OP. They are given by

$$(ax^2+by^2+cz^2)(a\alpha^2+b\beta^2+c\gamma^2)=(a\alpha x+b\beta y+c\gamma z)^2.$$

The diametral plane of OP is also the polar plane of P.

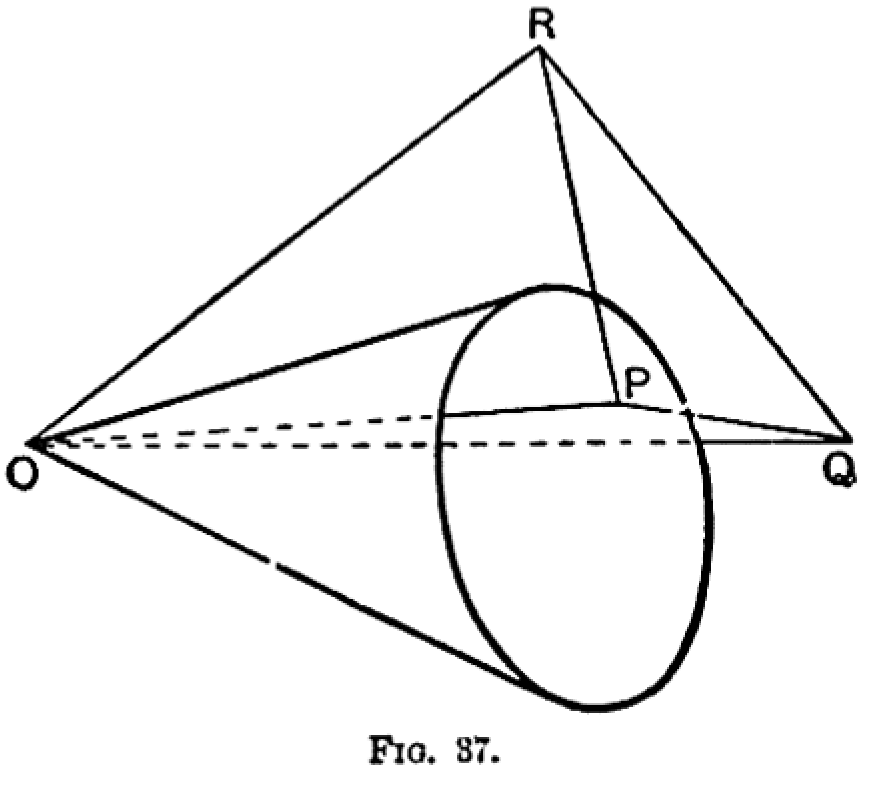

FIG. 37.

Conjugate diameters. Let OP, OQ, OR, (fig. 37), any three conjugate diameters of the cone, meet any plane in P, Q, R. The plane meets the cone in a conic, and QR is the locus of the harmonic conjugates of P with respect to the points in which any secant through P cuts the conic; *i.e.* QR is the polar of P with respect to the conic. Similarly, RP and PQ are the polars of Q and R, and therefore the triangle PQR is self-polar with respect to the conic. Conversely, if PQR is any triangle self-polar with respect to the conic in which the plane PQR cuts the cone, OP, OQ, OR are conjugate diameters of the cone. For the polar plane of P passes through the line QR and through the vertex, and therefore OQR is the polar plane of P, or the diametral plane of OP; and similarly, ORP, OPQ are the diametral planes of OQ and OR.

Ex. 1. The locus of the asymptotic lines drawn from O to the conicoid $ax^2+by^2+cz^2=1$ is the **asymptotic cone** $ax^2+by^2+cz^2=0$.

Ex. 2. The hyperboloids

$$x^2/a^2+y^2/b^2-z^2/c^2=1, \quad -x^2/a^2-y^2/b^2+z^2/c^2=1$$

have the same asymptotic cone. Draw a figure shewing the cone and the two hyperboloids.

Ex. 3. The section of a hyperboloid by a plane which is parallel to a tangent plane of the asymptotic cone is a parabola.

Ex. 4. If a plane through the origin cuts the cones

$$ax^2+by^2+cz^2=0, \quad \alpha x^2+\beta y^2+\gamma z^2=0$$

in lines which form a harmonic pencil, it touches the cone

$$\frac{x^2}{b\gamma+c\beta}+\frac{y^2}{c\alpha+a\gamma}+\frac{z^2}{a\beta+b\alpha}=0.$$

For the following examples the axes are rectangular.

Ex. 5. Planes which cut $ax^2+by^2+cz^2=0$ in perpendicular generators touch

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0.$$

Ex. 6. The lines of intersection of pairs of tangent planes to $ax^2+by^2+cz^2=0$ which touch along perpendicular generators lie on the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

Ex. 7. Perpendicular tangent planes to $ax^2+by^2+cz^2=0$ intersect in generators of the cone

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

Ex. 8. If the cone $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ passes through a set of conjugate diameters of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1, \quad \text{then} \quad Aa^2+Bb^2+Cc^2=0.$$

Ex. 9. If three conjugate diameters of an ellipsoid meet the director sphere in P, Q, R, the plane PQR touches the ellipsoid.

Ex. 10. Find the equation to the normal plane (*i.e.* at right angles to the tangent plane) of the cone $ax^2+by^2+cz^2=0$ which passes through the generator $x/l=y/m=z/n$.

Ans. $\Sigma\frac{(b-c)x}{l}=0.$

Ex. 11. Lines drawn through the origin at right angles to normal planes of the cone $ax^2+by^2+cz^2=0$ generate the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

Ex. 12. If the two cones $ax^2+by^2+cz^2=0$, $\alpha x^2+\beta y^2+\gamma z^2=0$ have each sets of three mutually perpendicular generators, any two planes which pass through their four common generators are at right angles.

THE PARABOLOIDS.

79. The locus of the equation

$$(1)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}, \qquad (2)\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation (1) represents the surface generated by the variable ellipse $z=k$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}$. This ellipse is imaginary unless k and c have the same sign, hence the centre of the ellipse lies on OZ if $c>0$ and on OZ′ if $c<0$. The sections of the surface by planes parallel to the coordinate planes YOZ, ZOX are parabolas. Fig. 38 shews the form and position of the surface for a positive value of c. The surface is the **elliptic paraboloid.**

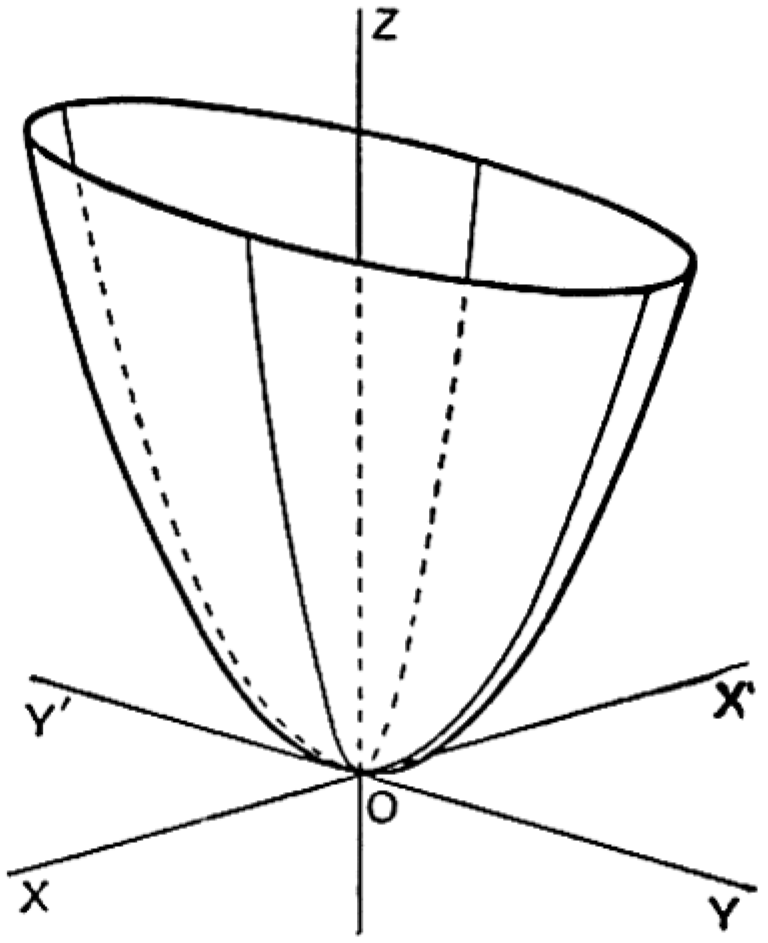

FIG. 38.

The equation (2) represents the surface generated by the variable hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$, $z=k$. The hyperbola is real for all real values of k, and its centre passes in turn through every point on Z′Z. When $k=0$ the hyperbola degenerates into the two lines $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, z=0$. The sections of the surface by the planes $z=k$, $z=-k$ project on the plane XOY into conjugate hyperbolas whose asymptotes are $z=0$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$. The sections by planes parallel to YOZ, ZOX are parabolas. The surface is the **hyperbolic paraboloid**, and fig. 39 shews the form and position of the surface for a negative value of c.

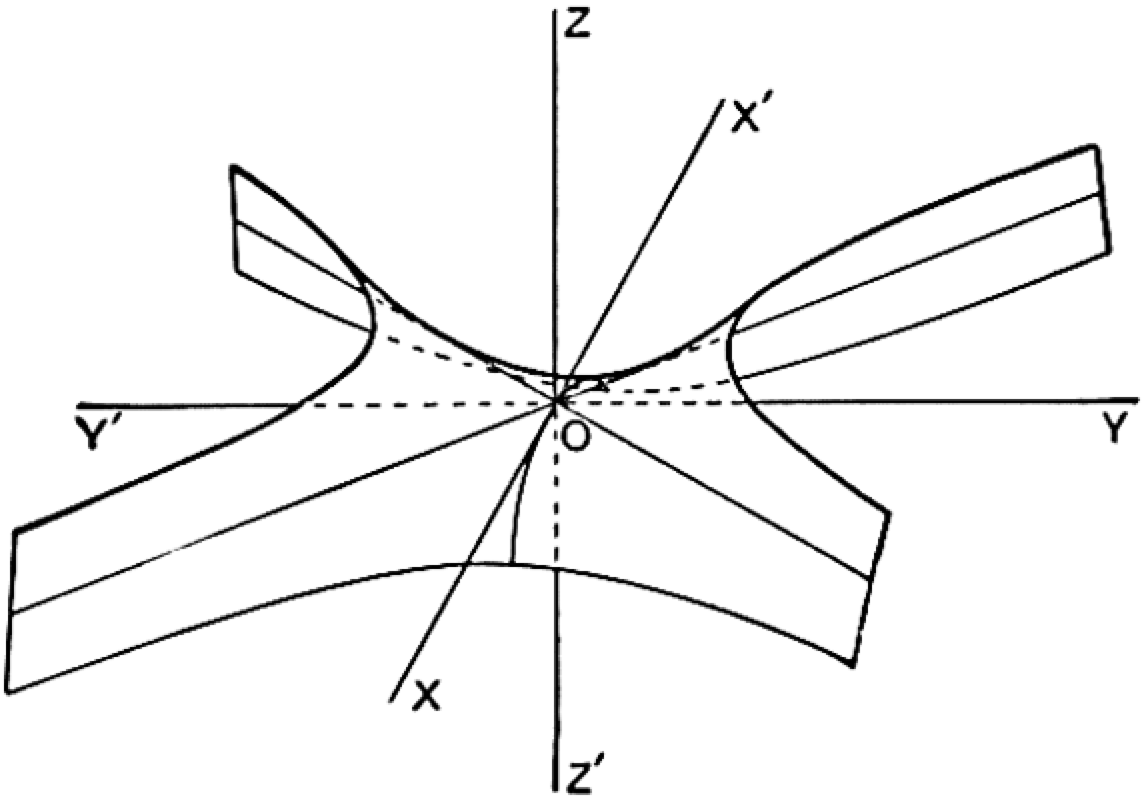

FIG. 39.

80. Conjugate diametral planes. An equation of the form

$$ax^2+by^2=2z$$

represents a paraboloid. Any line in the plane XOY which passes through the origin meets the surface in two coincident points, and hence the plane XOY is the tangent plane at the origin. The planes YOZ, ZOX bisect chords parallel to OX and OY respectively. Each is therefore parallel to the chords bisected by the other. Such pairs of planes are called **conjugate diametral planes** of the paraboloid

81. Diameters. If A is the point (α, β, γ), and the equations to a line through A are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (=r),$$

the distances from A to the points of intersection of the line and the paraboloid are given by

$$r^2(al^2+bm^2)+2r(a\alpha l+b\beta m-n)+a\alpha^2+b\beta^2-2\gamma=0. \ldots(1)$$

If $l=m=0$, one value of r is infinite, and therefore a line parallel to the z-axis meets the paraboloid in one point at an infinite distance, and in a point P whose distance from A is given by

$$r=\frac{a\alpha^2+b\beta^2-2\gamma}{-2(a\alpha l+b\beta m-n)}=\frac{a\alpha^2+b\beta^2-2\gamma}{2n}.$$

Such a line is called a **diameter**, and P is the extremity of the diameter.

Hence $ax^2+by^2=2z$ represents a paraboloid, referred to a tangent plane, and two conjugate diametral planes through the point of contact, as coordinate planes. One of the coordinate axes is the diameter through the point of contact. If the axes are rectangular, so that the tangent plane at O is at right angles to the diameter through O, O is the **vertex** of the paraboloid, and the diameter through O is the **axis**. The coordinate planes YOZ, ZOX are then **principal planes.**

Ex. What surface is represented by the equation $xy=2cz$?

82. Tangent planes. We find, as in § 67, the equation to the tangent plane at the point (α, β, γ) on the paraboloid,

$$a\alpha x+b\beta y=z+\gamma.$$

If $lx+my+nz=p$ is a tangent plane and (α, β, γ) is the point of contact,

$$\alpha=\frac{-l}{an}, \quad \beta=\frac{-m}{bn}, \quad \gamma=\frac{-p}{n},$$

and therefore
$$\frac{l^2}{a}+\frac{m^2}{b}+2np=0.$$

Hence $2n(lx+my+nz)+\frac{l^2}{a}+\frac{m^2}{b}=0$ is the equation to the tangent plane to the paraboloid which is parallel to the plane $lx+my+nz=0$.

If the axes are rectangular and

$$2n_r(l_rx+m_ry+n_rz)+\frac{l_r^2}{a}+\frac{m_r^2}{b}=0, \quad (r=1, 2, 3),$$

represent three mutually perpendicular tangent planes, we have, by addition,

$$2z+\frac{1}{a}+\frac{1}{b}=0,$$

and therefore the locus of the point of intersection of three mutually perpendicular tangent planes is a plane at right angles to the axis of the paraboloid.

Ex. 1. Shew that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$, and find the coordinates of the point of contact.

Ans. (8, 9, 5).

Ex. 2. Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ intersect in a straight line lying in the plane $x=0$. Shew that the line touches the parabola

$$x=0, \quad y^2=(a+b)(2z+a).$$

Ex. 3. Shew that the locus of the tangents from a point (α, β, γ) to the paraboloid $ax^2+by^2=2z$ is given by

$$(ax^2+by^2-2z)(a\alpha^2+b\beta^2-2\gamma)=(a\alpha x+b\beta y-z-\gamma)^2.$$

Ex. 4. Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid.

Ans. $ab(x^2+y^2)-2(a+b)z-1=0$.

83. Diametral planes. If a line OP has equations $x/l=y/m=z/n$, the diametral plane of OP, *i.e.* the locus of the mid-points of chords parallel to OP, is given by $alx+bmy-n=0$. Hence all diametral planes are parallel to the axis of the paraboloid, and conversely any plane parallel to the axis is a diametral plane. If OQ, whose equations are $x/l'=y/m'=z/n'$, is parallel to the diametral plane of OP,

$$all'+bmm'=0. \qquad \text{............ ....(1)}$$

Hence OP is parallel to the diametral plane of OQ, and the diametral planes of OP and OQ are conjugate.

Equation (1) is the condition that the lines $alx+bmy=0$, $al'x+bm'y=0$, in the plane $z=k$, should be conjugate diameters of the conic $ax^2+by^2=2k$. Hence any plane meets a pair of conjugate diametral planes of a paraboloid in lines which are parallel to conjugate diameters of the conic in which the plane meets the surface.

Ex. 1. The locus of the centres of a system of parallel plane sections of a paraboloid is a diameter.

Ex. 2. The plane $3x+4y=1$ is a diametral plane of the paraboloid $5x^2+6y^2=2z$. Find the equations to the chord through (3, 4, 5) which it bisects. *Ans.* $\frac{x-3}{9}=\frac{y-4}{10}=\frac{z-5}{15}$.

Ex. 3. Any diametral plane cuts the paraboloid in a parabola, and parallel diametral planes cut it in equal parabolas.

84. The normals. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ represents an elliptic paraboloid, referred to rectangular axes, the normal at (x', y', z') has for equations

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}}=\frac{z-z'}{-1}.$$

If this normal passes through a given point (α, β, γ),

$$\frac{\alpha-x'}{\frac{x'}{a^2}}=\frac{\beta-y'}{\frac{y'}{b^2}}=\frac{\gamma-z'}{-1}=\lambda, \quad \text{say.}$$

Therefore $x'=\frac{a^2\alpha}{a^2+\lambda}, \quad y'=\frac{b^2\beta}{b^2+\lambda}, \quad z'=\gamma+\lambda,$

and $$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda).$$

This equation gives five values of λ, and hence there are five points on the paraboloid the normals at which pass through a given point.

Ex. 1. Prove that the feet of the normals from any point to the paraboloid lie on a cubic curve.

Ex. 2. Prove that the normals from (α, β, γ) to the paraboloid lie on the cone

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Ex. 3. Prove that the cubic curve lies on this cone.

Ex. 4. Prove that the perpendicular from (α, β, γ) to its polar plane lies on the cone.

Ex. 5. In general three normals can be drawn from a given point to the paraboloid of revolution $x^2+y^2=2az$, but if the point lies on the surface $27a(x^2+y^2)+8(a-z)^3=0$, two of the three normals coincide.

Ex. 6. Shew that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ lie on the sphere

$$x^2+y^2+z^2-z(a+\gamma)-\frac{y}{2\beta}(\alpha^2+\beta^2)=0.$$

Ex. 7. Shew that the centre of the circle through the feet of the three normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ is

$$\left(\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma+a}{2}\right).$$

*Examples IV.

1. Two asymptotic lines can be drawn from a point **P** to a conicoid $ax^2+by^2+cz^2=1$, and they are at right angles if **P** lies on the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

2. The lines in which the plane $lx+my+nz=0$ cuts the cone $\alpha x^2+\beta y^2+\gamma z^2=0$ are conjugate diameters of the ellipse in which it cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Prove that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lies on the cone

$$a^2(\beta b^2+\gamma c^2)x^2+b^2(\gamma c^2+\alpha a^2)y^2+c^2(\alpha a^2+\beta b^2)z^2=0.$$

3. **P** and **Q** are points on an ellipsoid. The normal at **P** meets the tangent plane at **Q** in **R**; the normal at **Q** meets the tangent plane at **P** in **S**. If the perpendiculars from the centre to the tangent planes at **P** and **Q** are p_1, p_2, prove that **PR** : **QS** $=p_2 : p_1$.

4. The line of intersection of the tangent planes at **P** and **Q**, points on $ax^2+by^2+cz^2=1$, passes through a fixed point **A**, (α, β, γ), and is parallel to the plane **XOY**. Shew that the locus of the mid-point of **PQ** is the conic in which the polar plane of **A** cuts the surface

$$ax^2+by^2+cz^2=z/\gamma.$$

5. Shew that the greatest value of the shortest distance between the axis of x and a normal to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is $b-c$.

6. Plane sections of an ellipsoid which have their centres on a given straight line are parallel to a fixed straight line and touch a parabolic cylinder.

7. **OP**, **OQ**, **OR** are conjugate diameters of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

At **Q** and **R** tangent lines are drawn parallel to **OP**, and p_1, p_2 are their distances from **O**. The perpendicular from **O** to the tangent plane at right angles to **OP** is p. Prove that

$$p^2+p_1{}^2+p_2{}^2=a^2+b^2+c^2.$$

8. Conjugate diameters of

$$a_1x^2+b_1y^2+c_1z^2=1 \quad \text{meet} \quad a_2x^2+b_2y^2+c_2z^2=1$$

in **P**, **Q**, **R**. Shew that the plane **PQR** touches the conicoid

$$a_3x^2+b_3y^2+c_3z^2=1,$$

where $$\frac{a_3}{a_1}=\frac{b_3}{b_1}=\frac{c_3}{c_1}=\frac{a_2}{a_1}+\frac{b_2}{b_1}+\frac{c_2}{c_1}.$$

9. The ellipsoid which has as conjugate diameters the three straight lines that bisect pairs of opposite edges of a tetrahedron touches the edges.

10. Shew that the projections of the normals to an ellipsoid at **P**, **Q**, **R**, the extremities of conjugate diameters on the plane **PQR**, are concurrent.

11. If through any given point (α, β, γ) perpendiculars are drawn to any three conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the plane through the feet of the perpendiculars passes through the fixed point

$$\left(\frac{a^2\alpha}{a^2+b^2+c^2},\quad \frac{b^2\beta}{a^2+b^2+c^2},\quad \frac{c^2\gamma}{a^2+b^2+c^2}\right).$$

12. If perpendiculars be drawn from any point **P** on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ to any three conjugate diametral planes, the plane through the feet of the perpendiculars meets the normal at **P** at a fixed point whose distance from **P** is

$$\frac{a^2b^2c^2}{p(b^2c^2+c^2a^2+a^2b^2)},$$

where p is the perpendicular from the centre to the tangent plane at **P**.

13. Find the locus of centres of sections of a conicoid that are at a constant distance from the centre.

14. Shew that the equations to the right circular cones that pass through the axes (which are rectangular) are $yz \pm zx \pm xy=0$.

Deduce that the lines through a given point **P**, which are perpendicular to their polars with respect to $x^2/a^2+y^2/b^2+z^2/c^2=1$, lie upon a right circular cone if **P** lies on one of the lines

$$(b^2-c^2)^2x^2=(c^2-a^2)^2y^2=(a^2-b^2)^2z^2.$$

15. Chords of a conicoid which are parallel to a given diameter and are such that the normals at their extremities intersect, lie in a fixed plane through the given diameter.

16. The perpendiculars from the origin to the faces of the tetrahedron whose vertices are the feet of the four normals from a point to the cone $ax^2+by^2+cz^2=0$, lie on the cone

$$a(b-c)^2x^2+b(c-a)^2y^2+c(a-b)^2z^2=0.$$

17. P, Q, R; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. Prove that the poles of the planes PQR, P′Q′R′ lie on the surface $\Sigma\left\{\dfrac{b^2z^2+c^2y^2}{b^2-c^2}(x^2-a^2)\right\}=0$.

18. The normals at P and P′, points of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

meet the plane XOY in A and A′ and make angles θ, θ' with PP′. Prove that $\mathrm{PA}\cos\theta+\mathrm{P'A'}\cos\theta'=0$.

19. The normals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ at all points of its intersection with $lyz+mzx+nxy=0$ intersect the line

$$\frac{a^2x}{l(a^2-b^2)(c^2-a^2)}=\frac{b^2y}{m(b^2-c^2)(a^2-b^2)}=\frac{c^2z}{n(c^2-a^2)(b^2-c^2)}.$$

20. Shew that the points on an ellipsoid the normals at which intersect a given straight line lie on the curve of intersection of the ellipsoid and a conicoid.

21. The normals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ at points of its intersection with $x/a+y/b+z/c=1$ lie on the surface

$$\Sigma\left(\frac{ab\,xy+bc\,yz+ca\,zx}{b(a^2-c^2)y+c(a^2-b^2)z}\right)=1.$$

22. Prove that two normals to $ax^2+by^2+cz^2=1$ lie in the plane $lx+my+nz=p$, and that they are at right angles if

$$abcp^2\Sigma\{a(b+c)l^2\}=\Sigma\{a^2(b-c)^2m^2n^2\}.$$

23. The locus of a point, the sum of the squares of whose normal distances from the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is constant, $(=k^2)$, is

$$6x^2+6y^2+6z^2-2\Sigma\left\{\frac{a^2x^2(b^2+c^2-2a^2)}{(a^2-b^2)(c^2-a^2)}\right\}+2a^2+2b^2+2c^2=k^2.$$

24. If the feet of the six normals from (α, β, γ) are

$$(x_r, y_r, z_r), \quad (r=1, 2, \dots 6),$$

prove that $$a^2\alpha\Sigma\left(\frac{1}{x_r}\right)+b^2\beta\Sigma\left(\frac{1}{y_r}\right)+c^2\gamma\Sigma\left(\frac{1}{z_r}\right)=0.$$

25. If the feet of three of the normals from P to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ lie in the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$, the feet of the other three lie in the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}+1=0$, and P lies on the line

$$a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z.$$

26. If A, B are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, the pair of tangent planes at the points where AB cuts the conicoid $S \equiv ax^2+by^2+cz^2-1=0$ is given by

$$S_2P_1^2-2P_1P_2P_{12}+S_1P_2^2=0,$$

and the pair of tangent planes that intersect in AB, by

$$S(S_1S_2-P^2_{12})-S_2P_1^2+2P_1P_2P_{12}-S_1P_2^2=0,$$

where

$$S_1 \equiv a\alpha_1^2+b\beta_1^2+c\gamma_1^2-1, \text{ etc.};$$
$$P_1 \equiv a\alpha_1x+b\beta_1y+c\gamma_1z-1, \text{ etc.};$$
$$P_{12} \equiv a\alpha_1\alpha_2+b\beta_1\beta_2+c\gamma_1\gamma_2-1.$$

27. If P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2), R, (x_3, y_3, z_3) are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $OP=r_1$, $OQ=r_2$, $OR=r_3$, prove that the equation to the sphere OPQR can be written

$$x^2+y^2+z^2-r_1^2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}\right)-r_2^2\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}\right)$$
$$-r_3^2\left(\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}\right)=0,$$

and prove that the locus of the centres of spheres through the origin and the extremities of three equal conjugate semi-diameters is

$$12(a^2x^2+b^2y^2+c^2z^2)=(a^2+b^2+c^2)^2.$$

CHAPTER VIII.

THE AXES OF A PLANE SECTION OF A CONICOID.

85. We have proved, (§ 54, Exs. 2, 3), that every plane section of a conicoid is a conic, and that parallel plane sections are similar and similarly situated conics. We now proceed to find equations to determine the magnitudes and directions of the axes of a given plane section of a given conicoid.

General method for determining the axes. If the lengths of the axes of a conic are 2α and 2β, and $\alpha > r > \beta$, the conic has two diameters of length $2r$, and they are equally inclined to the axes. If $r = \alpha$ or β, the two diameters of length $2r$ coincide with an axis. Hence to find the axes of the conic in which a given plane cuts a conicoid, we first form the equation to a cone whose vertex is the centre, C, of the conic and which has as generators the lines of length r which can be drawn in the plane from C to the conicoid. The lines of section of this cone and the given plane are the semi-diameters of length r of the conic. If $2r$ is the length of an axis, these are coincident, or the plane touches the cone, the generator of contact being the axis. The condition of tangency gives an equation which determines r; the comparison of the equations of the given plane and a tangent plane to the cone leads to the direction-cosines of the generator of contact.

86. Axes of the section of a central conicoid by a plane through the centre. Let the equations, referred to rectangular axes, of the conicoid and plane be

$$ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = 0.$$

The centre of the conicoid is also the centre of the section. If λ, μ, ν are the direction-cosines of a semi-diameter of the conicoid of length r, the point $(\lambda r, \mu r, \nu r)$ is on the conicoid. Therefore

$$a\lambda^2+b\mu^2+c\nu^2=\frac{1}{r^2}=\frac{\lambda^2+\mu^2+\nu^2}{r^2}.$$

Hence the semi-diameters of the conicoid of length r are generators of the cone

$$x^2(a-1/r^2)+y^2(b-1/r^2)+z^2(c-1/r^2)=0. \quad\text{.........(1)}$$

The lines of section of the cone and plane are the semi-diameters of the conic of length r. Hence, if r is the length of either semi-axis of the conic in which the plane $lx+my+nz=0$ cuts the conicoid, the plane touches the cone, and therefore

$$\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0, \quad\text{................(2)}$$

$$\text{or } r^4(bcl^2+cam^2+abn^2)-r^2\{(b+c)l^2+(c+a)m^2+(a+b)n^2\}$$
$$+(l^2+m^2+n^2)=0.$$

The roots of this quadratic in r^2 give the squares of the semi-axes of the section.

If $2r$ is the length of an axis and λ, μ, ν are the direction-cosines, the given plane touches the cone (1) along the line $x/\lambda=y/\mu=z/\nu$, and therefore is represented by the equation

$$\lambda x(a-1/r^2)+\mu y(b-1/r^2)+\nu z(c-1/r^2)=0.$$

$$\text{Therefore}\quad \frac{\lambda(ar^2-1)}{l}=\frac{\mu(br^2-1)}{m}=\frac{\nu(cr^2-1)}{n}. \quad\text{...........(3)}$$

These determine the direction-cosines of the axis of length $2r$.

Since the extremities of the semi-diameters of length r of the conicoid lie upon the sphere $x^2+y^2+z^2=r^2$, the equation of the cone through them may be obtained by making the equation to the conicoid homogeneous by means of the equation to the sphere. Thus the cone is

$$ax^2+by^2+cz^2=\frac{x^2+y^2+z^2}{r^2},$$

which is another form of equation (1).

Ex. 1. Prove that the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ lie on the cone

$$(b-c)\frac{l}{x}+(c-a)\frac{m}{y}+(a-b)\frac{n}{z}=0.$$

(From equations (3) we deduce that

$$(b-c)\frac{l}{\lambda}+(c-a)\frac{m}{\mu}+(a-b)\frac{n}{\nu}=0.)$$

Ex. 2. Prove that the cone of Ex. 1 passes through the normal to the plane of section and the diameter to which the plane of section is diametral plane. Prove also that the cone passes through two sets of conjugate diameters of the conicoid. (Cf. Ex. 6, § 77.)

Ex. 3. Find the lengths of the semi-axes of the conics given by

$$\text{(i)}\quad 3x^2+2y^2+6z^2=1, \qquad x+y+z=0\,;$$

$$\text{(ii)}\quad 2x^2+y^2-z^2=1, \quad 3x+4y+5z=0.$$

Ans. (i) ·64, ·45; (ii) 3·08, ·76.

Ex. 4. Prove that the equation of the conic

$$x^2+2y^2-2z^2=1, \quad 3x-2y-z=0,$$

referred to its principal axes, is approximately

$$1{\cdot}70x^2-1{\cdot}77y^2=1.$$

Ex. 5. Prove that the lengths of the semi-axes of the section of the ellipsoid of revolution $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$, by the plane $lx+my+nz=0$, are

$$a, \quad ac(l^2+m^2+n^2)^{\frac{1}{2}}\{a^2(l^2+m^2)+c^2n^2\}^{-\frac{1}{2}},$$

and that the equations to the axes are

$$\frac{x}{m}=\frac{y}{-l}=\frac{z}{0}\,; \quad \frac{x}{nl}=\frac{y}{mn}=\frac{z}{-(l^2+m^2)}.$$

Ex. 6. Prove that the area of the section of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

by the plane $lx+my+nz=0$ is $\dfrac{\pi abc}{p}$, where p is the perpendicular from the centre to the tangent plane which is parallel to the given plane.

Ex. 7. The section of the conicoid $ax^2+by^2+cz^2=1$ by a tangent plane to the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$$

is a rectangular hyperbola.

Ex. 8. The section of a hyperboloid of one sheet by a tangent plane to the asymptotic cone is two parallel straight lines. What is the corresponding section of the hyperboloid of two sheets which has the same asymptotic cone?

Ex. 9. Central sections of an ellipsoid whose area is constant envelope a cone of the second degree.

Ex. 10. If A_1, A_2, A_3 are the areas of three mutually perpendicular central sections of an ellipsoid, $A_1^{-2}+A_2^{-2}+A_3^{-2}$ is constant.

Ex. 11. One of the axes of each section of the ellipsoid $\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$ by a tangent plane to the cone $y^2+z^2=x^2$ lies on the cone

$$(x^2+y^2)^2=z^2(x^2-y^2).$$

What is the nature of the section of this cone by a plane parallel to the plane XOY? Sketch the form of the cone.

Ex. 12. Prove that the axes of sections of the conicoid

$$ax^2+by^2+cz^2=1$$

which pass through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lie on the cone

$$\frac{b-c}{x}(mz-ny)+\frac{c-a}{y}(nx-lz)+\frac{a-b}{z}(ly-mx)=0.$$

87. Axes of any section of a central conicoid. Let the equations, referred to rectangular axes, of the conicoid and plane be $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

Then if C, (α, β, γ) is the centre of the section, the plane is also represented by the equation

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

Therefore $$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}. \quad\text{..............(1)}$$

Hence $$a\alpha^2+b\beta^2+c\gamma^2=\frac{p^2}{l^2/a+m^2/b+n^2/c}=\frac{p^2}{p_0^2}, \text{ say.}$$

The equation to the conicoid referred to parallel axes through C is

$$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)+a\alpha^2+b\beta^2+c\gamma^2-1=0,$$

or $$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)-k^2=0,$$

where $$k^2\equiv 1-\frac{p^2}{p_0^2}.$$

The equation to the plane is now $lx+my+nz=0$.

If λ, μ, ν are the direction-cosines of a line of length r drawn from C to the conicoid,

$$r^2(a\lambda^2+b\mu^2+c\nu^2)+2r(a\alpha\lambda+b\beta\mu+c\gamma\nu)-k^2=0.$$

If the line lies in the given plane

$$l\lambda+m\mu+n\nu=0,$$

and therefore, by (1), $a\alpha\lambda+b\beta\mu+c\gamma\nu=0.$

Hence $r^2(a\lambda^2+b\mu^2+c\nu^2)-k^2(\lambda^2+\mu^2+\nu^2)=0$,
and therefore the semi-diameters of the section of length r lie on the cone

$$x^2\left(\frac{ar^2}{k^2}-1\right)+y^2\left(\frac{br^2}{k^2}-1\right)+z^2\left(\frac{cr^2}{k^2}-1\right)=0.$$

If r is the length of either semi-axis of the section, the plane touches the cone. Therefore

$$\frac{l^2}{\frac{ar^2}{k^2}-1}+\frac{m^2}{\frac{br^2}{k^2}-1}+\frac{n^2}{\frac{cr^2}{k^2}-1}=0. \quad\ldots\ldots\ldots\ldots\ldots(2)$$

And, as in §86, the direction-cosines of the axis of length $2r$ are given by

$$\frac{\lambda\left(\frac{ar^2}{k^2}-1\right)}{l}=\frac{\mu\left(\frac{br^2}{k^2}-1\right)}{m}=\frac{\nu\left(\frac{cr^2}{k^2}-1\right)}{n}. \quad\ldots\ldots\ldots\ldots(3)$$

Comparing these equations with equations (2) and (3) of §86, we see that if α and β are the lengths of the semi-axes of the section by the plane $lx+my+nz=0$, the semi-axes of the section by the plane $lx+my+nz=p$ are $k\alpha$ and $k\beta$, or
$$\alpha\sqrt{1-\frac{p^2}{p_0^{\,2}}},\quad \beta\sqrt{1-\frac{p^2}{p_0^{\,2}}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$
and that the corresponding axes are parallel. We thus have another proof for central surfaces of the proposition that parallel plane sections are similar and similarly situated conics.

From equations (4) it follows that if A, A_0 are the areas of the sections of a conicoid by a given plane and the parallel plane through the centre, $A=A_0\left(1-\frac{p^2}{p_0^{\,2}}\right)$, where p, p_0 are the perpendiculars from the centre to the given plane and the parallel tangent plane. Thus the area of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane

$$lx+my+nz=p$$

is
$$\frac{\pi abc(l^2+m^2+n^2)^{\frac{1}{2}}}{(a^2l^2+b^2m^2+c^2n^2)^{\frac{1}{2}}}\left\{1-\frac{p^2}{a^2l^2+b^2m^2+c^2n^2}\right\}.$$

The student should note that the equation to the cone through *all* the lines of length r drawn from **C** to the conicoid would be obtained by making the equation

$$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)-k^2=0$$

homogeneous by means of the equation $x^2+y^2+z^2=r^2$. It would be of the *fourth* degree, while for our purpose we require a cone of the second degree. The cone chosen passes through the lines of length r which lie in the given plane, and these lines alone need be considered.

Ex. 1. **OP** is a given semi-diameter of a conicoid and **OA**$(=\alpha)$, **OB**$(=\beta)$, are the principal semi-axes of the section of the diametral plane of **OP**. A plane parallel to **AOB** meets **OP** in **C**. Prove that the principal axes of the section of the conicoid by this plane are $\alpha\sqrt{1-\text{OC}^2/\text{OP}^2}$, $\beta\sqrt{1-\text{OC}^2/\text{OP}^2}$, and deduce equations (4), § 87.

(Take **OP**, **OA**, **OB**, as coordinate axes.)

Ex. 2. Find the coordinates of the centre and the lengths of the semi-axes of the section of the ellipsoid $3x^2+3y^2+6z^2=10$ by the plane $x+y+z=1$.

Ans. $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$; $\sqrt{\frac{44}{15}}, \frac{\sqrt{44}}{5}$.

Ex. 3. If **OP**, **OQ**, **OR** are conjugate semi-diameters of an ellipsoid prove that the area of the section of the ellipsoid by the plane **PQR** is two-thirds the area of the parallel central section.

Ex. 4. Find the area of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $x/a+y/b+z/c=1$.

Ans. $\frac{2\pi}{3\sqrt{3}}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}$.

Ex. 5. Find the locus of the centres of sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose area is constant, $(=\pi k^2)$.

Ans. $a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^2=k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$.

Ex. 6. Prove that tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$ which cut $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$ in ellipses of constant area πk^2 have their points of contact on the surface $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^4}{4a^2b^2c^2}$

Ex. 7. Prove that the axes of the section whose centre is **P** are the straight lines in which the plane of section cuts the cone containing the normals from **P**.

Ex. 8. Find the lengths of the semi-axes of the sections of the surface $4yz+5zx-5xy=8$ by the planes (i) $x+y-z=0$, (ii) $2x+y-z=0$.

Ans. (i) 2, $\sqrt{3}$. (ii) 2, 2.

Ex. 9. Prove that the axes of the section of

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ are given by

$$r^4(\mathrm{A}l^2\ldots+2\mathrm{F}mn\ldots)+r^2\{f(l,\,m,\,n)-(a+b+c)(l^2+m^2+n^2)\}$$
$$+(l^2+m^2+n^2)=0,$$

where $\mathrm{A}\equiv bc-f^2$, etc.

Prove also that the axes are the lines in which the plane cuts the cone

$$(mz-ny)\frac{\partial f}{\partial x}+(nx-lz)\frac{\partial f}{\partial y}+(ly-mx)\frac{\partial f}{\partial z}=0.$$

Ex. 10. Prove that the axes of the section of the cone

$$ax^2+by^2+cz^2=0$$

by the plane $lx+my+nz=p$ are given by

$$\frac{l^2}{ap_0{}^2r^2+p^2}+\frac{m^2}{bp_0{}^2r^2+p^2}+\frac{n^2}{cp_0{}^2r^2+p^2}=0,$$

where $p_0{}^2\equiv\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$.

88. Axes of a given section of a paraboloid. If the equations to the plane and the paraboloid are

$$lx+my+nz=p,\quad ax^2+by^2=2z,$$

the centre of the section, $(\alpha,\,\beta,\,\gamma)$, is given by

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{-1}{n}=\frac{a\alpha^2+b\beta^2-\gamma}{p}.$$

Whence $$a\alpha^2+b\beta^2-2\gamma=\frac{l^2/a+m^2/b+2np}{-n^2}=\frac{p_0{}^2}{-n^2},\text{ say.}$$

Changing the origin to $(\alpha,\,\beta,\,\gamma)$ and proceeding as in § 87, we find that $\lambda,\,\mu,\,\nu$, the direction-cosines of a semi-diameter of length r of the section, satisfy the equation

$$n^2r^2(a\lambda^2+b\mu^2)-p_0{}^2(\lambda^2+\mu^2+\nu^2)=0.$$

The semi-diameters are therefore the lines in which the plane cuts the cone

$$x^2(an^2r^2-p_0{}^2)+y^2(bn^2r^2-p_0{}^2)-z^2p_0{}^2=0.$$

Hence the lengths of the axes are given by

$$\frac{l^2}{an^2r^2-p_0{}^2}+\frac{m^2}{bn^2r^2-p_0{}^2}-\frac{n^2}{p_0{}^2}=0,$$

or

$$abn^6r^4-n^2r^2p_0{}^2\{(a+b)n^2+am^2+bl^2\}+p_0{}^4(l^2+m^2+n^2)=0$$

and the direction cosines by

$$\frac{\lambda(an^2r^2-p_0{}^2)}{l}=\frac{\mu(bn^2r^2-p_0{}^2)}{m}=\frac{\nu p_0{}^2}{-n}.$$

Ex. 1. Find the lengths of the semi-axes of the section of the paraboloid $2x^2+y^2=z$ by the plane $x+2y+z=4$. *Ans.* 5·28, 1·68.

Ex. 2. A plane section through the vertex of the paraboloid of revolution $x^2+y^2=2az$ makes an angle θ with the axis of the surface. Prove that its principal semi-axes are $a \cot\theta \operatorname{cosec}\theta$, $a\cot\theta$.

Ex. 3. Prove that the semi-axes of the section of the paraboloid $xy=az$ by the plane $lx+my+nz=0$ are given by

$$n^6r^4-4a^2r^2l^2m^2n^2-4a^4l^2m^2(l^2+m^2+n^2)=0.$$

Ex. 4. Find the locus of the centres of sections of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ which are of constant area πk^2.

Ans. $a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z\right)^2=k^4.$

Ex. 5. Given that the radius of curvature at a point P of a conic whose centre is C is equal to $\mathrm{CD}^3/\alpha\beta$, where α and β are the semi-axes and CD is the semi-diameter conjugate to CP, find the radius of curvature at the origin of the conic $ax^2+by^2=2z$, $lx+my+nz=0$.

Ans. $(l^2+m^2)^{\frac{3}{2}}(am^2+bl^2)^{-1}(l^2+m^2+n^2)^{-\frac{1}{2}}.$

Ex. 6. Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0.$$

CIRCULAR SECTIONS.

89. If $F=0$, the equation to a conicoid, can be thrown into the form $S+\lambda uv=0$, where $S=0$ is the equation to a sphere and $u=0$, $v=0$ represent planes, the common points of the conicoid and planes lie on the sphere, and therefore the sections of the conicoid by the planes are circles.

90. The circular sections of an ellipsoid. The equation, referred to rectangular axes, of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

can be written in the forms

$$\frac{x^2+y^2+z^2}{a^2}-1+y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0,$$

$$\frac{x^2+y^2+z^2}{b^2}-1+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\frac{x^2+y^2+z^2}{c^2}-1+x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0.$$

Hence the planes

$$\text{(i)}\ y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0,$$

$$\text{(ii)}\ z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\text{(iii)}\ x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0$$

cut the ellipsoid in circles of radii a, b, c respectively. If $a > b > c$, only the second of these equations gives real planes, and therefore the only real central circular sections of the ellipsoid pass through the mean axis, and are given by the equations

$$\frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0.$$

Since parallel plane sections are similar and similarly situated conics, the equations

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda,\quad \frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}=\mu$$

give circular sections for all values of λ and μ.

91. *Any two circular sections of an ellipsoid which are not parallel lie on a sphere.*

The equation $\quad k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+$

$$\left(\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\lambda\right)\left(\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\mu\right)=0$$

represents a conicoid which passes through the sections, and if $k=b^2$, the equation becomes

$$x^2+y^2+z^2-\frac{(\lambda+\mu)\sqrt{a^2-b^2}}{a}x+\frac{(\lambda-\mu)\sqrt{b^2-c^2}}{c}z+\lambda\mu-b^2=0,$$

which represents a sphere.

92. Circular sections of the hyperboloids. By the method of § 90, we deduce that the real central circular sections of the hyperboloids

$$\text{(i)}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \text{(ii)}\ \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad (a>b>c)$$

are given by

$$\text{(i) } \frac{y}{b}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{a^2+c^2}=0, \quad \text{(ii) } \frac{x}{a}\sqrt{a^2+b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0.$$

The radius of the central circular sections of the hyperboloid of one sheet is a. The planes given by

$$\frac{x}{a}\sqrt{a^2+b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$$

do not meet the hyperboloid of two sheets in any real points. They are the planes through the centre parallel to systems of planes which cut the surface in real circles.

Ex. 1. Prove that the section of the hyperboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ by the plane $\frac{x}{a}\sqrt{a^2+b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$ is real if $\lambda^2 > a^2+c^2$.

Ex. 2. Find the real central circular sections of the ellipsoid $x^2+2y^2+6z^2=8$. *Ans.* $x^2-4z^2=0$.

Ex. 3. Prove that the planes $2x+3z-5=0$, $2x-3z+7=0$ meet the hyperboloid $-x^2+3y^2+12z^2=75$ in circles which lie on the sphere

$$3x^2+3y^2+3z^2+4x+36z-110=0.$$

Ex. 4. Prove that the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$$

cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $b\sqrt{1-\frac{\lambda^2}{a^2-c^2}}$.

Ex. 5. Find the locus of the centres of spheres of constant radius k which cut the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in a pair of circles. (Use § 91.)

Ans. $y=0,\ \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1-\frac{k^2}{b^2}$.

Ex. 6. Chords of the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, are drawn so as to make equal angles with its axes, and on them as diameters circles are described whose planes are parallel to OZ. Prove that these circles generate the ellipsoid $2b^2x^2+2a^2y^2+(a^2+b^2)z^2=2a^2b^2$.

93. Circular sections of any central conicoid.

An equation of the form

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

represents a central conicoid. It may be written

$$f(x, y, z)-\lambda(x^2+y^2+z^2)+\lambda\left(x^2+y^2+z^2-\frac{1}{\lambda}\right)=0.$$

Hence if $\quad f(x, y, z)-\lambda(x^2+y^2+z^2)=0$

represents a pair of planes, these planes cut the conicoid in circles. For a pair of planes

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0.$$

This equation gives three values of λ. It can be shown that these are always real, and that only the mean value gives real planes.[1]

Ex. 1. Find the real central circular sections of the conicoid

$$3x^2+5y^2+3z^2+2xz=4.$$

The equation may be written

$$3x^2+5y^2+3z^2+2xz-\lambda(x^2+y^2+z^2)+\lambda(x^2+y^2+z^2)-4=0.$$

If $3x^2+5y^2+3z^2+2xz-\lambda(x^2+y^2+z^2)=0$ represents a pair of planes, $\lambda^3-11\lambda^2+38\lambda-40=0$, or $\lambda=2$, 4, or 5. For these values of λ the equation to the planes becomes

$$(x+z)^2+3y^2=0, \quad (x-z)^2-y^2=0, \quad x^2-xz+z^2=0.$$

The real circular sections correspond therefore to $\lambda=4$ and have equations

$$x-z+y=0, \quad x-z-y=0.$$

Ex. 2. Find the equations to the real central circular sections of the conicoids,

(i) $5y^2-8z^2+18yz-14zx-10xy+27=0$,

(ii) $2x^2+5y^2+2z^2-yz-4zx-xy+4=0$,

(iii) $6x^2+13y^2+6z^2-10yz+4zx-10xy=1$

Ans. (i) $(x-2y-5z)(3x-4y+z)=0$,

(ii) $(x+y+z)(2x-y+2z)=0$,

(iii) $2(x+z)^2-10y(x+z)+9y^2=0$.

Ex. 3. Find the equations to the circular sections of the conicoid

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)+1=0.$$

Ans. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\lambda, \quad ax+by+cz=\mu.$

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[1] See the author's *Coordinate Geometry of Three Dimensions*, §§ 144, 145.

Ex. 4. Find the conditions that the equations

$$f(x, y, z)=1, \quad lx+my+nz=0$$

should determine a circle.

The equation $f(x, y, z)-\lambda(x^2+y^2+z^2)=0$ is to represent two planes, one of which is the given plane. Therefore

$$f(x, y, z)-\lambda(x^2+y^2+z^2)\equiv(lx+my+nz)\left\{(a-\lambda)\frac{x}{l}+(b-\lambda)\frac{y}{m}+(c-\lambda)\frac{z}{n}\right\}.$$

Whence, comparing coefficients of yz, zx, xy, we obtain

$$\lambda=\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2}.$$

(We assume here that l, m, n are all different from zero. If $l=0$, the conditions become $(\lambda=a)$, $g=h=0$, $(c-a)m^2-2fmn+(b-a)n^2=0$.)

94. Circular sections of the paraboloids. The equation $ax^2+by^2=2z$ may be written in the forms,

$$a\left(x^2+y^2+z^2-\frac{2z}{a}\right)-y^2(a-b)-az^2=0,$$

$$b\left(x^2+y^2+z^2-\frac{2z}{b}\right)-x^2(b-a)-bz^2=0,$$

$$ax^2+by^2-(0\cdot x^2+0\cdot y^2+0\cdot z^2+2z)=0.$$

Hence if $a>b>0$, $x^2(a-b)=bz^2$ represents real planes which meet the paraboloid in circles, and the systems of circular sections are given by

$$x\sqrt{a-b}+z\sqrt{b}=\lambda, \quad x\sqrt{a-b}-z\sqrt{b}=\mu.$$

If, however, a or b is negative, the only real planes are those given by $ax^2+by^2=0$. The equation

$$0\cdot x^2+0\cdot y^2+0\cdot z^2+2z=0$$

is the limiting form of

$$kx^2+ky^2+k\left(z+\frac{1}{k}\right)^2=\frac{1}{k}$$

as k tends to zero, and therefore the sphere containing the circular sections is in this case of infinite radius, and the circular sections are circles of infinite radius, *i.e.* straight lines. They are the straight lines in which the plane $z=0$ cuts the surface, (§ 79).

Ex. 1. Find the circular sections of the paraboloid $x^2+10z^2=2y$.
Ans. $y\pm 3z=\lambda$.

Ex. 2. Find the radius of the circle in which the plane $7x+2z=5$ cuts the paraboloid $53x^2+4y^2=8z$. *Ans.* $r^2=\frac{314}{53}$.

95. Umbilics. The centres of a series of parallel plane sections of a conicoid lie upon a diameter of the conicoid and the tangent plane at an extremity of the diameter is parallel to the plane sections. If, therefore, P and P′ (fig. 40) are the extremities of the diameter which passes through the centres of a system of circular sections of an ellipsoid, the tangent planes at P and P′ are the limiting positions of the cutting planes, and P and P′ may be regarded as circular sections of zero radius. A circular section of zero radius is called an **umbilic**. It is evident from the form of the hyperboloid of one sheet that the smallest closed section is the principal elliptic section and that the surface has therefore no real umbilics.

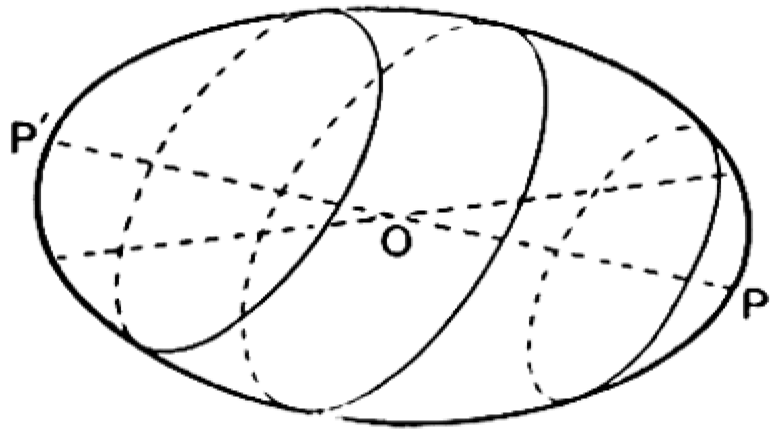

FIG. 40.

To find the umbilics of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

If P, (ξ, η, ζ) is an umbilic, the diametral plane of OP is a central circular section. Therefore the equations

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0, \quad \frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$$

represent the same plane. Hence

$$\frac{\xi/a}{\sqrt{a^2-b^2}}=\frac{\eta/b}{0}=\frac{\zeta/c}{\pm\sqrt{b^2-c^2}}$$

$$=\frac{1}{\pm\sqrt{a^2-c^2}}, \quad \text{since} \quad \frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1,$$

and therefore

$$\xi=\frac{\pm a\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}, \quad \eta=0, \quad \zeta=\frac{\pm c\sqrt{b^2-c^2}}{\sqrt{a^2-c^2}}.$$

These give the coordinates of the four umbilics.

The umbilics of the hyperboloid of two sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

are real and given by

$$\xi=\frac{\pm a\sqrt{a^2+b^2}}{\sqrt{a^2+c^2}},\quad \eta=0,\quad \zeta=\frac{\pm c\sqrt{b^2-c^2}}{\sqrt{a^2+c^2}}.$$

Ex. 1. Prove that the umbilics of the ellipsoid lie on the sphere

$$x^2+y^2+z^2=a^2-b^2+c^2.$$

Ex. 2. Prove that the perpendicular distance from the centre to the tangent plane at an umbilic of the ellipsoid is ac/b.

Ex. 3. Prove that the central circular sections of the conicoid $(a-b)x^2+ay^2+(a+b)z^2=1$ are at right angles and that the umbilics are given by $x=\pm\sqrt{\frac{a+b}{2a(a-b)}}$, $y=0$, $z=\pm\sqrt{\frac{a-b}{2a(a+b)}}$.

Ex. 4. Prove that the umbilics of the conicoid $\frac{x^2}{a+b}+\frac{y^2}{a}+\frac{z^2}{a-b}=1$ are the extremities of the equal conjugate diameters of the ellipse

$$y=0,\quad \frac{x^2}{a+b}+\frac{z^2}{a-b}=1.$$

Ex. 5. Prove that the umbilics of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$, $a>b$, are $\left(0,\ \pm b\sqrt{a^2-b^2},\ \frac{a^2-b^2}{2}\right)$.

Ex. 6. Deduce the coordinates of the umbilics of the ellipsoid from the result of Ex. 4, § 92.

*Examples V.

1. Prove that if λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2 are the direction-cosines of the axes of any plane section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,

$$\frac{\lambda_1\lambda_2}{a^2(b^2-c^2)}=\frac{\mu_1\mu_2}{b^2(c^2-a^2)}=\frac{\nu_1\nu_2}{c^2(a^2-b^2)}.$$

2. If $\Delta_1, \Delta_2, \Delta_3$; $\delta_1, \delta_2, \delta_3$ are the areas of the sections of the ellipsoids $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, by three conjugate diametral planes of the former,

$$\frac{\Delta_1^2}{\delta_1^2}+\frac{\Delta_2^2}{\delta_2^2}+\frac{\Delta_3^2}{\delta_3^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right).$$

3. If A_1, A_2, A_3 are the areas of the sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

by the diametral planes of three mutually perpendicular semi-diameters of lengths r_1, r_2, r_3,

$$\frac{A_1^2}{r_1^2}+\frac{A_2^2}{r_2^2}+\frac{A_3^2}{r_3^2}=\pi^2\left(\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}\right).$$

4. Through a given point (α, β, γ) planes are drawn parallel to three conjugate diametral planes of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Shew that the sum of the ratios of the areas of the sections by these planes to the areas of the parallel diametral planes is $3-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}$.

5. Prove that the areas of the sections of greatest and least area of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which pass through the fixed line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ are $\frac{\pi abc}{r_1}$, $\frac{\pi abc}{r_2}$, where r_1, r_2 are the axes of the section by the plane $\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}=0$.

6. Prove that the systems of circular sections of the cone

$$ax^2+by^2+cz^2=0, \quad a>b>c,$$

are given by $x\sqrt{a-b}\pm z\sqrt{b-c}=\lambda$, and that these also give circular sections of the cone $(a+\mu)x^2+(b+\mu)y^2+(c+\mu)z^2=0$.

7. Any tangent plane to a cone cuts the cyclic planes in lines equally inclined to the generator of contact.

8. Any pair of tangent planes to the cone $ax^2+by^2+cz^2=0$ cuts the cyclic planes $x\sqrt{a-b}\pm z\sqrt{b-c}=0$ in lines which lie upon a right circular cone whose axis is at right angles to the plane of contact.

9. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts a series of central conicoids whose principal planes are the coordinate planes in rectangular hyperbolas. Shew that the pole of the plane with respect to the conicoids lies on a cone whose section by the given plane is a circle.

10. OP, OQ, OR are conjugate diameters of an ellipsoid, axes a, b, c, and S is the foot of the perpendicular from O to the plane PQR. Shew that the cone whose vertex is S and base is the section of the ellipsoid by the diametral plane parallel to the plane PQR has constant volume $\pi abc/3\sqrt{3}$.

11. If two cones with the same vertex have the same systems of circular sections, their common tangent planes touch them along perpendicular generators.

12. The normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at all points of a central circular section are parallel to a plane that makes an angle $\cos^{-1}\frac{ac}{b\sqrt{a^2-b^2+c^2}}$ with the section.

13. If r_1, r_2 are the axes of a central section of an ellipsoid, and θ_1, θ_2 the angles between the section and the circular sections,

$$\sin\theta_1 . \sin\theta_2=\frac{a^2c^2}{a^2-c^2}\left(\frac{1}{r_1{}^2}\sim\frac{1}{r_2{}^2}\right),$$

where a and c are the greatest and least axes of the ellipsoid.

14. Through a fixed point which is the pole of a circular section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are drawn planes cutting the surface in rectangular hyperbolas. Shew that the centres of these hyperbolas lie on a fixed circle whose plane is parallel to one system of circular sections.

15. The locus of the centres of sections of the cone $ax^2+by^2+cz^2=0$, such that the sum of the squares of their axes is constant, $(=k^2)$, is the conicoid

$$a\left(\frac{1}{b}+\frac{1}{c}\right)x^2+b\left(\frac{1}{c}+\frac{1}{a}\right)y^2+c\left(\frac{1}{a}+\frac{1}{b}\right)z^2+k^2=0.$$

16. The area of a central section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is constant. Shew that the axes of the section lie on the cone

$$\Sigma\frac{a^2-p^2}{a^4}\left(\frac{c^2-a^2}{c^2}\frac{z}{y}-\frac{a^2-b^2}{b^2}\frac{y}{z}\right)^2=0,$$

where p is the distance from the centre of a tangent plane parallel to any of the planes of section.

17. Prove that the tangents at the vertices to the parabolic sections of the conicoid $ax^2+by^2+cz^2=1$ are parallel to generators of the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

18. Prove that the normals to central sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which are of given eccentricity e, lie on the cone

$$\begin{aligned}a^2b^2c^2(e^2-2)^2(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)\\=(1-e^2)\{a^2(b^2+c^2)x^2+b^2(c^2+a^2)y^2+c^2(a^2+b^2)z^2\}^2\end{aligned}$$

Find the locus of the centres of sections of eccentricity e.

19. Prove that the normal at any point P of an ellipsoid is an axis of some plane section of the ellipsoid. If the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and P is the point (x', y', z'), shew that the length of the axis is

$$p^{-3}\left(\frac{x'^2}{a^6}+\frac{y'^2}{b^6}+\frac{z'^2}{c^6}\right)^{-1},$$

where p is the perpendicular from the centre to the tangent plane at P.

20. The normal section of an enveloping cylinder of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ has a given area πk^2. Prove that the plane of contact of the cylinder and ellipsoid touches the cone

$$\frac{x^2}{a^4(b^2c^2-k^4)}+\frac{y^2}{b^4(c^2a^2-k^4)}+\frac{z^2}{c^4(a^2b^2-k^4)}=0.$$

21. Prove that the locus of the foci of parabolic sections of the paraboloid $ax^2+by^2=2z$ is

$$ab(2z-ax^2-by^2)(ax^2+by^2)=a^2x^2+b^2y^2.$$

22. Prove that the equation to a conicoid referred to the tangent plane and normal at an umbilic as xy-plane and z-axis is

$$a(x^2+y^2)+cz^2+2fyz+2gzx+2wz=0.$$

If a variable sphere be described to touch a given conicoid at an umbilic, it meets the conicoid in a circle whose plane moves parallel to itself as the radius of the sphere varies.

23. If through the centre of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ a perpendicular is drawn to any central section and lengths equal to the axes of the section are marked off along the perpendicular, the locus of their extremities is given by

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

where $r^2\equiv x^2+y^2+z^2$. (The locus is the *Wave Surface.*)

24. Prove that the asymptotes of sections of the conicoid

$$ax^2+by^2+cz^2=1$$

which pass through the line $x=h$, $y=0$ lie on the surface

$$\{ax(x-h)+by^2\}^2+cz^2\{a(x-h)^2+by^2\}=0.$$

25. If the section of the cone whose vertex is **P**, (α, β, γ) and base $z=0$, $ax^2+by^2=1$, by the plane $x=0$ is a circle, then **P** lies on the conic $y=0$, $ax^2-bz^2=1$, and the section of the cone by the plane

$$(a-b)\gamma x-2a\alpha z=0$$

is also a circle.

CHAPTER IX.

GENERATING LINES.

96.[1] **Ruled surfaces.** In cones and cylinders we have examples of surfaces which are generated by a moving straight line. Such surfaces are called **ruled surfaces.** We shall now prove that the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces.

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, which represents a hyperboloid of one sheet, may be written,

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

Whence it appears that the hyperboloid is the locus of the straight lines whose equations are

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right);\ \ldots\ldots\ldots(1)$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right);\ \ldots\ldots\ldots(2)$$

where λ and μ are variable parameters. It is obviously impossible to assign values to λ and μ so that the equations (1) become identical with the equations (2). Hence the equations give two distinct systems of lines, no member of one coinciding with any member of the other. As λ assumes in turn all real values the line given by the equations (1) moves so as to completely generate the hyperboloid. Similarly, the line given by the equations (2) moves, as μ varies, so as to generate the hyperboloid. The hyperboloid of one sheet is therefore a ruled surface and can be generated in two ways by the motion of a straight line. (See fig. 41.)

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[1] See Appendix, p. iv.

In like manner the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$$

which represents a hyperbolic paraboloid, may be written

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z.$$

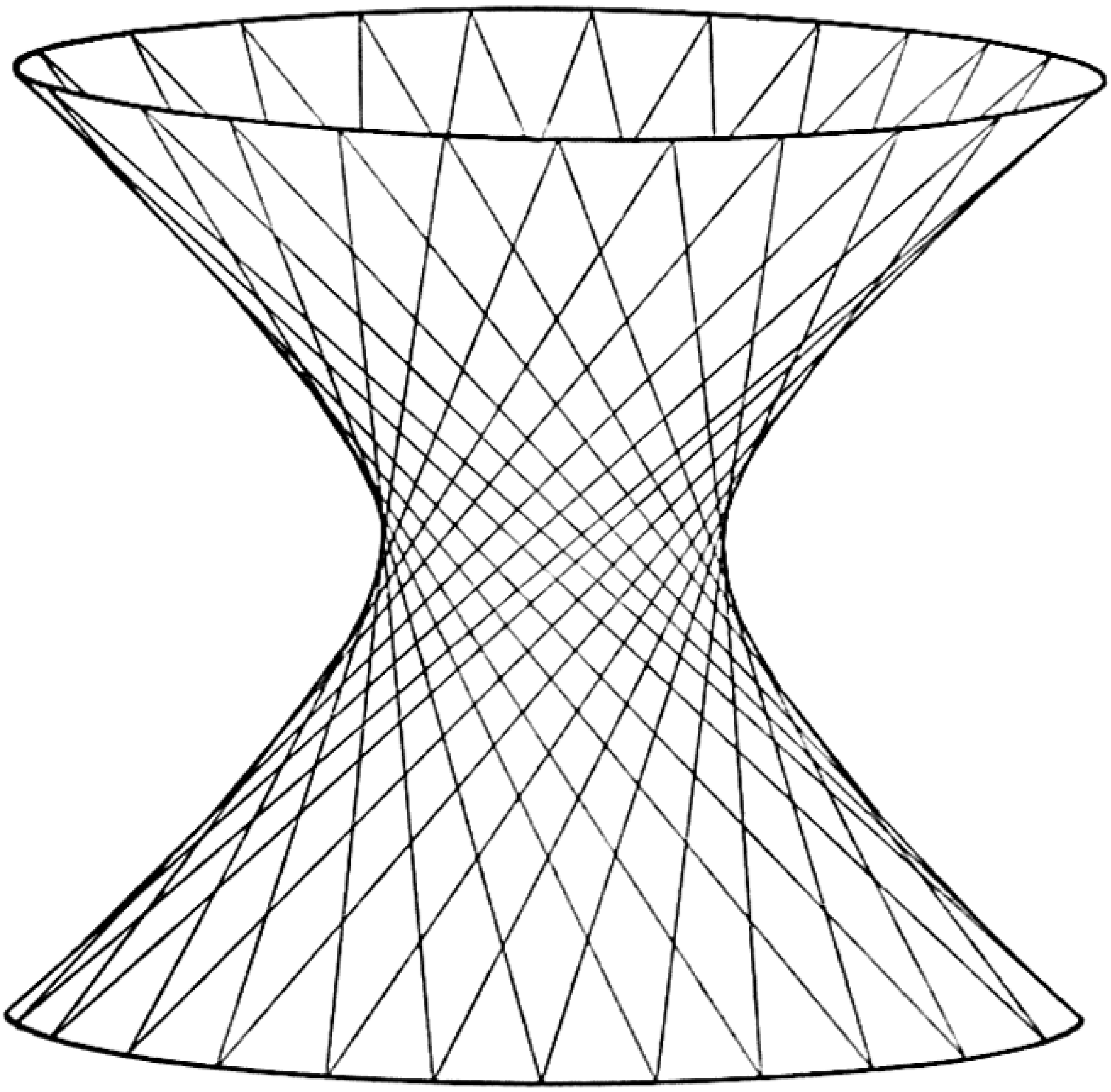

FIG. 41.

Whence it is evident that the paraboloid is the locus of either of the variable lines given by

$$\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda},\quad \frac{x}{a}-\frac{y}{b}=2\lambda;$$

$$\frac{x}{a}-\frac{y}{b}=\frac{z}{\mu},\quad \frac{x}{a}+\frac{y}{b}=2\mu.$$

The hyperbolic paraboloid is therefore a ruled surface and can be generated in two ways by the motion of a straight line. (See fig. 42.) The generating lines are parallel to one of the fixed planes $\frac{x}{a}\pm\frac{y}{b}=0$.

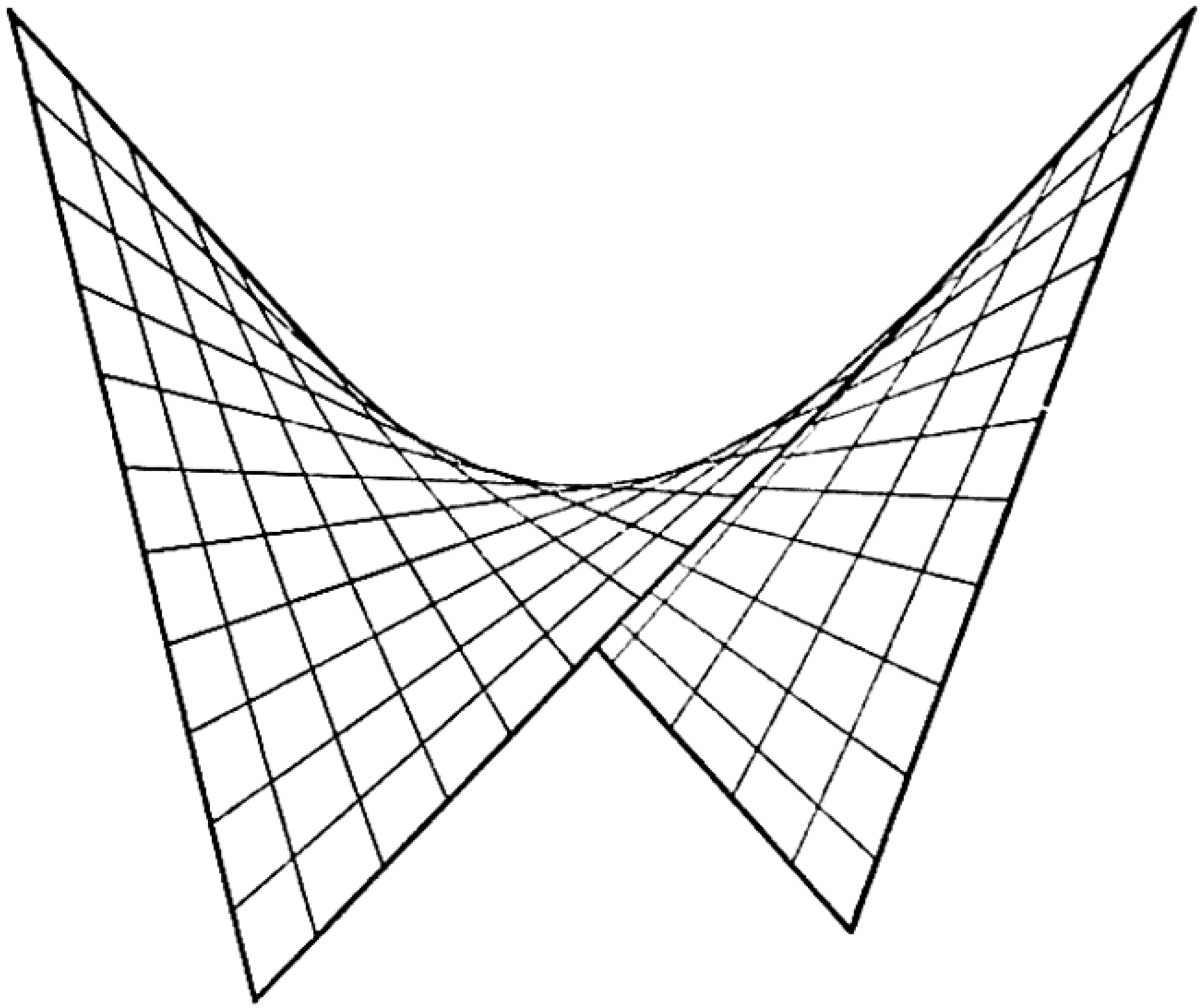

FIG. 42

Ex. 1. **CP**, **CQ** are any conjugate diameters of the ellipse

$$x^2/a^2+y^2/b^2=1, \quad z=c.$$

C′P′, **C′Q′** are the conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$, drawn in the same directions as **CP** and **CQ**. Prove that the hyperboloid $\frac{2x^2}{a^2}+\frac{2y^2}{b^2}-\frac{z^2}{c^2}=1$ is generated by either **PQ′** or **P′Q**.

Ex. 2. A point, "m," on the parabola $y=0$, $cx^2=2a^2z$, is $(2am, 0, 2cm^2)$, and a point, "n," on the parabola $x=0$, $cy^2=-2b^2z$, is $(0, 2bn, -2cn^2)$. Find the locus of the lines joining the points for which, (i) $m=n$, (ii) $m=-n$.

Ans. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$.

97. Section of a ruled surface by the tangent plane at a point.

Since a hyperboloid of one sheet or a hyperbolic paraboloid is generated completely by each of two systems of straight lines, there pass through any point **P**, (fig. 43), of the surface, two generating lines, one from each system. Each of these meets the surface at **P** in, at least, two coincident points, and therefore the lines lie in the tangent plane at P. The tangent plane at P is therefore the plane through the generators which pass through P. But any plane section of the surface is a conic, and therefore the section of the surface by the tangent plane at P is the conic composed of the two generating lines through P.

It follows that if a straight line AB lies wholly on the conicoid it must belong to one of the systems of generating lines. For AB meets any generating line PQ in some point P, and AB and PQ both lie in the tangent plane at P. But the section of the surface by the tangent plane at P consists of the two generators through P, and therefore AB must be one of the generators.

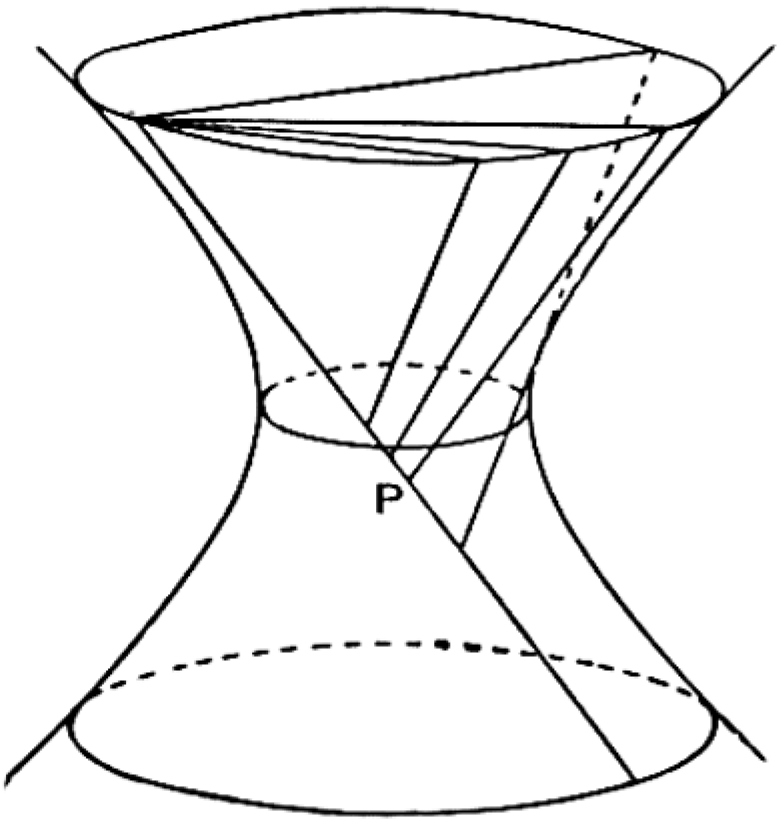

FIG. 43.

Again any plane through a generating line is the tangent plane at some point of the generating line. For the locus of points common to the surface and plane is a conic, and the generating line is obviously part of the locus. The locus must therefore consist of two straight lines, or the plane must pass through the given generating line and a second generating line which meets it. It is therefore the tangent plane at the point of intersection.

The intersection of a cone or cylinder with a tangent plane consists of two coincident generators. The ruled conicoids can therefore be divided into two classes according as the generators in which any tangent plane meets them are distinct or coincident. If the generators are distinct the tangent planes at different points of a given generator are different, (see fig. 43). If the generators are coincident, the same plane touches the surface at all points of a given generator.

98. *If three points of a straight line lie on a conicoid the straight line lies wholly on the conicoid.*

The coordinates of any point on the line through (α, β, γ), whose direction-ratios are l, m, n, are $\alpha+lr$, $\beta+mr$, $\gamma+nr$. The condition that this point should lie on the conicoid $F(x, y, z)=0$ may be written, since $F(x, y, z)$ is of the second degree, in the form

$$\mathsf{A}r^2+2\mathsf{B}r+\mathsf{C}=0.$$

If three points of the line lie on the conicoid, this equation is satisfied by three values of r, and therefore $\mathsf{A}=\mathsf{B}=\mathsf{C}=0$. The equation is therefore satisfied by all values of r, and every point of the line lies on the conicoid.

99. *To find the conditions that a given straight line should be a generator of a given conicoid.*

Let the equations to the conicoid and line be

$$ax^2+by^2+cz^2=1,$$

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

The point on the line, $(\alpha+lr, \beta+mr, \gamma+nr)$, lies on the conicoid if

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+a\alpha^2+b\beta^2+c\gamma^2-1=0.$$

If this equation is an identity, the line lies wholly on the conicoid, and is a generator of the conicoid. The required conditions are therefore

$$a\alpha^2+b\beta^2+c\gamma^2=1, \quad\text{..................}(1)$$
$$a\alpha l+b\beta m+c\gamma n=0, \quad\text{..................}(2)$$
$$al^2+bm^2+cn^2=0. \quad\text{..................}(3)$$

Equation (1) is the condition that (α, β, γ) should lie on the surface; equation (2) shews that a generating line must lie in the tangent plane at any point (α, β, γ) on it; and from (3) it follows that the parallels through the centre to the generating lines generate the asymptotic cone

$$ax^2+by^2+cz^2=0.$$

The three equations (1), (2), (3) shew that through any point (α, β, γ) of a central conicoid two straight lines can be drawn to lie wholly on the conicoid, the direction-ratios of these lines being given by equations (2) and (3). By Lagrange's identity, we have

$$(al^2+bm^2)(a\alpha^2+b\beta^2)-(a\alpha l+b\beta m)^2 \equiv ab(\alpha m-\beta l)^2;$$

whence, by (1), (2), (3),

$$-cn^2=ab(\alpha m-\beta l)^2. \quad \text{.....................(4)}$$

The values of $l:m:n$ are therefore real only if ab and c have opposite signs, which can only be the case if two of the quantities a, b, c are positive and one is negative. The only ruled central conicoid is therefore the hyperboloid of one sheet. From equations (2) and (4) we deduce the direction-ratios of the generators through (α, β, γ),

$$\frac{l}{\pm b\beta\sqrt{\frac{-c}{ab}}+c\alpha\gamma}=\frac{m}{\mp a\alpha\sqrt{\frac{-c}{ab}}+c\beta\gamma}=\frac{n}{-(a\alpha^2+b\beta^2)}.$$

Similarly, the conditions that the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

should be a generator of the paraboloid $ax^2+by^2=2z$ are,

$$a\alpha^2+b\beta^2=2\gamma, \quad \text{.....................(1)}$$

$$a\alpha l+b\beta m-n=0, \quad \text{.....................(2)}$$

$$al^2+bm^2=0. \quad \text{.....................(3)}$$

Equation (3) is satisfied by real values of $l:m$ only if a and b have opposite signs. The only ruled paraboloid is therefore the hyperbolic paraboloid. The direction-ratios of the generating lines through (α, β, γ) are given by

$$\frac{l}{\sqrt{\frac{1}{a}}}=\frac{m}{\pm\sqrt{\frac{-1}{b}}}=\frac{n}{\alpha\sqrt{a}\pm\beta\sqrt{-b}}.$$

The following examples should be solved in two ways, (i) by factorising the equation to the surface as in § 96, (ii) by means of the conditions in § 99.

Ex. 1. Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the points $(2, 3, -4)$, $(2, -1, \frac{4}{3})$.

Ans. $\frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}$; $\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4}$;
$\frac{x-2}{0}=\frac{y+1}{3}=\frac{z-\frac{4}{3}}{-4}$; $\frac{x-2}{3}=\frac{y+1}{6}=\frac{z-\frac{4}{3}}{10}$.

Ex. 2. Find the equations to the generating lines of the hyperboloid $yz+2zx+3xy+6=0$ which pass through the point $(-1, 0, 3)$.

$$[yz+2zx+3xy+6\equiv(y+2)(z+3)+(2z+3y)(x-1).]$$

Ans. $x=-1$, $z=3$; $\frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$.

Ex. 3. Find the equations to the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$.

Ans. $\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}$.

Ex. 4. Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(1, 1, 1)$.

Ans. $\frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}$; $\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{1}$.

THE SYSTEMS OF GENERATING LINES.

100. We shall call the systems of generating lines of the hyperboloid of one sheet which are given by the equations

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right); \quad \text{..........(1)}$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right), \quad \text{..........(2)}$$

the λ-system and μ-system, respectively.

101. *No two generators of the same system intersect.*

For the equations (1) and

$$\frac{x}{a}+\frac{z}{c}=\lambda'\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda'}\left(1-\frac{y}{b}\right)$$

lead to $\frac{x}{a}+\frac{z}{c}=0,\quad \frac{x}{a}-\frac{z}{c}=0,\quad 1+\frac{y}{b}=0,\quad 1-\frac{y}{b}=0,$

which are obviously inconsistent.

Otherwise, if P and Q are any points on any generator of the μ-system and the generators of the λ-system through P and Q intersect at R, then the plane PQR meets the hyperboloid in the sides of a triangle. This is impossible, since no plane section of a conicoid is of higher degree than the second.

102. *Any generator of the λ-system intersects any generator of the μ-system.*

From the equations (1) and (2),

$$\frac{\frac{x}{a}+\frac{z}{c}}{\lambda}=\frac{\frac{x}{a}-\frac{z}{c}}{\mu}=\frac{1-\frac{y}{b}}{\lambda\mu}=\frac{1+\frac{y}{b}}{1}.$$

Whence, adding and subtracting numerators and denominators,

$$\frac{x}{a}=\frac{\lambda+\mu}{1+\lambda\mu},\quad \frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu},\quad \frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}.$$

These determine the point of intersection.

The equations

$$\frac{x}{a}+\frac{z}{c}-\lambda\left(1+\frac{y}{b}\right)+k\left\{\frac{x}{a}-\frac{z}{c}-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\right\}=0,\ \ldots\ldots(3)$$

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)+k'\left\{\frac{x}{a}+\frac{z}{c}-\frac{1}{\mu}\left(1-\frac{y}{b}\right)\right\}=0\ \ \ldots\ldots(4)$$

both reduce to

$$\frac{x}{a}(\lambda+\mu)+\frac{y}{b}(1-\lambda\mu)-\frac{z}{c}(\lambda-\mu)=1+\lambda\mu,\ \ldots\ldots\ldots(5)$$

if $k=1/k'=\lambda/\mu$. But equation (5) represents the tangent plane at the point of intersection of the generators. Hence the plane through two intersecting generators is the tangent plane at their common point. (Cf. § 97.)

If, in equation (3), k is given, the equation represents a given plane through the generator. But the equation reduces to equation (5) if $\mu=\lambda/k$. Hence any plane through a generating line is a tangent plane.

Ex. Discuss the intersection of the λ-generator through P with the μ-generator through P′ when P and P′ are the extremities of a diameter of the principal elliptic section.

103. Perpendicular generators. *To find the locus of the points of intersection of perpendicular generators.*

The direction-cosines of the λ- and μ-generators are given by, (§ 42),

$$\frac{l/a}{\lambda^2-1}=\frac{m/b}{2\lambda}=\frac{n/c}{\lambda^2+1}, \qquad \frac{l/a}{\mu^2-1}=\frac{m/b}{2\mu}=\frac{n/c}{-(\mu^2+1)}.$$

The condition that the generators should be at right angles is

$$a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(\lambda^2+1)(\mu^2+1)=0,$$

which may be written

$$a^2(\lambda+\mu)^2+b^2(1-\lambda\mu)^2+c^2(\lambda-\mu)^2=(a^2+b^2-c^2)(1+\lambda\mu)^2,$$

and shews that their point of intersection

$$\left\{\frac{a(\lambda+\mu)}{1+\lambda\mu},\ \frac{b(1-\lambda\mu)}{1+\lambda\mu},\ \frac{c(\lambda-\mu)}{1+\lambda\mu}\right\}$$

lies on the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2.$$

The locus is therefore the curve of intersection of the hyperboloid and the director sphere.

Or if PQ, PR are perpendicular generators and PN is normal at P, by § 102 the planes PQR, PNQ, PNR are mutually perpendicular tangent planes, and therefore P lies on the director sphere.

104. *The projections of the generators of a hyperboloid on a principal plane are tangents to the section of the hyperboloid by the principal plane.*

The projections of the λ- and μ-generators on the plane XOY are given by

$$z=0,\quad \frac{2x}{a}=\lambda\left(1+\frac{y}{b}\right)+\frac{1}{\lambda}\left(1-\frac{y}{b}\right);$$

$$z=0,\quad \frac{2x}{a}=\mu\left(1+\frac{y}{b}\right)+\frac{1}{\mu}\left(1-\frac{y}{b}\right),$$

which may be written

$$z=0,\quad \lambda^2\left(1+\frac{y}{b}\right)-\frac{2\lambda x}{a}+\left(1-\frac{y}{b}\right)=0,\quad \text{etc.}$$

Whence the envelope of the projections is the ellipse

$$z=0,\quad \frac{x^2}{a^2}=1-\frac{y^2}{b^2}.$$

Similarly, the projections on the planes YOZ, ZOX touch the corresponding principal sections.

The above equations to the projections are identical if $\lambda=\mu$. Hence equal values of the parameters give two generators which project into the same tangent to the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$. The point of intersection, P, of the generators given by $\lambda=\mu=t$ is, by § 102,

$$\left(a\frac{2t}{1+t^2},\quad b\frac{1-t^2}{1+t^2},\quad 0\right),$$

i.e. is $(a\cos\alpha,\ b\sin\alpha,\ 0)$, where $t\equiv\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$. P is therefore the point on the principal elliptic section whose eccentric angle is α, and the generators project into the tangent

$$z=0,\quad \frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1,\quad \text{(fig. 44).}$$

From § 103, the direction-cosines of the λ-generator are proportional to

$$a\frac{\lambda^2-1}{\lambda^2+1},\quad b\frac{2\lambda}{\lambda^2+1}\quad c,$$

or, since $\lambda=\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$, $a\sin\alpha,\ -b\cos\alpha,\ -c.$

Therefore the equations[1] to the λ-generator through P are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{-c}.$$

Similarly, the equations to the μ-generator are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}.$$

Ex. Prove that the generators given by $\lambda=t$, $\mu=-1/t$ are parallel, and that they meet the principal elliptic section in the extremities of a diameter.

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[1] See Appendix, p. iv.

105. Let P and Q, (fig. 44), be the points on the principal elliptic section whose eccentric angles are α and β. Then the λ-generator through P intersects the μ-generator through Q at the point, R,

$$\left(a\frac{\lambda+\mu}{1+\lambda\mu},\quad b\frac{1-\lambda\mu}{1+\lambda\mu},\quad c\frac{\lambda-\mu}{1+\lambda\mu}\right),$$

where $\lambda\equiv\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$ and $\mu\equiv\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

Whence the coordinates of R are

$$a\frac{\cos\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}},\quad b\frac{\sin\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}},\quad c\tan\frac{\beta-\alpha}{2}.$$

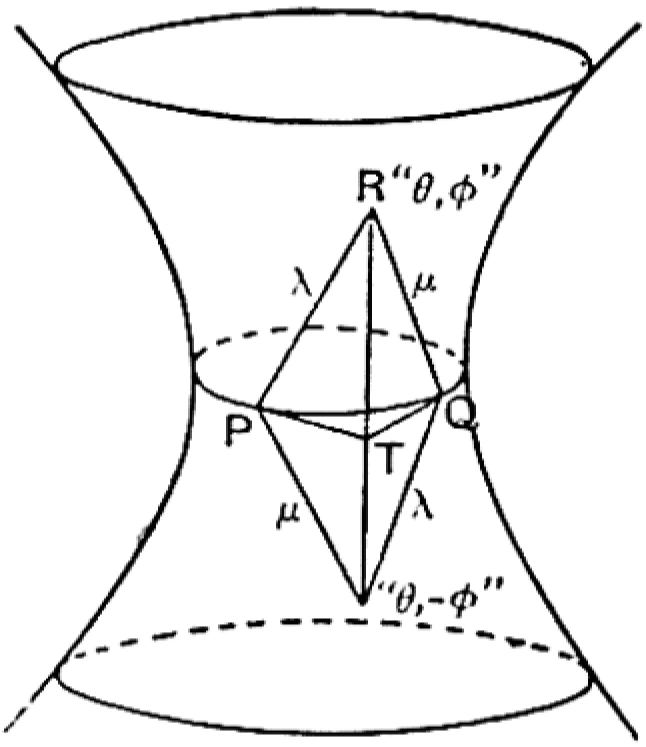

FIG. 44.

Now the coordinates of any point on the hyperboloid can be expressed in the form

$$a\cos\theta\sec\phi,\quad b\sin\theta\sec\phi,\quad c\tan\phi;$$

therefore if R is the point "θ, ϕ,"

$$\theta=\frac{\beta+\alpha}{2}\quad\text{and}\quad\phi=\frac{\beta-\alpha}{2},$$

or

$$\theta+\phi=\beta,\quad\theta-\phi=\alpha.$$

Similarly, it may be shewn that the μ-generator through P intersects the λ-generator through Q at the point "$\theta, -\phi$."

Suppose now that P remains fixed while Q varies, so that α is constant and β variable. R then moves along a fixed generator of the λ-system, and in any position $\theta-\phi=\alpha$. Hence for points on a generator of the λ-system $\theta-\phi$ is constant. Similarly, by supposing Q to remain fixed and P to vary, we can prove that for points on a given generator of the μ-system $\theta+\phi$ is constant.

Ex. 1. If R is "θ, ϕ," (fig. 44), shew that the equations to PQ are $z=0$, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\cos\phi$, and deduce that $\theta-\phi=\alpha$, $\theta+\phi=\beta$.

Ex. 2. The equations to the generating lines through "θ, ϕ" are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c}.$$

Ex. 3. If $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$ is a point on the generating line

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right),$$

prove that $\tan\frac{\theta-\phi}{2}=\frac{1-\lambda}{1+\lambda}$, and hence shew that for points of a given generator of the λ-system $\theta-\phi$ is constant.

Ex. 4. Prove that the equations

$$\frac{x}{a}=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)},\quad \frac{y}{b}=\frac{\cos\theta\sin\phi}{\cos(\theta+\phi)},\quad \frac{z}{c}=\frac{\sin\theta\cos\phi}{\cos(\theta+\phi)}$$

determine a hyperboloid of one sheet, that θ is constant for points on a given generator of one system, and that ϕ is constant for points on a given generator of the other system.

(The equation to the surface is $\frac{x^2}{a^2}-\frac{4yz}{bc}=1$.)

Ex. 5. Find the locus of R if P and Q are the extremities of conjugate diameters of the principal elliptic section.

We have $\theta-\phi=\alpha$, $\theta+\phi=\alpha\pm\frac{\pi}{2}$, whence $\phi=\pm\frac{\pi}{4}$, and R lies in one of the planes $z=\pm c$.

Ex. 6. Prove also that $RP^2+RQ^2=a^2+b^2+2c^2$.

Ex. 7. If A and A′ are the extremities of the major axis of the principal elliptic section, and any generator meets two generators of the same system through A and A′ respectively in P and P′, prove that $AP.A'P'=b^2+c^2$.

Ex. 8. Prove also that the planes APP′, A′PP′ cut either of the real central circular sections in perpendicular lines.

Ex. 9. If four generators of the hyperboloid form a skew quadrilateral whose vertices are "θ_r, ϕ_r," $r=1, 2, 3, 4$, prove that

$$\theta_1+\theta_3=\theta_2+\theta_4,\quad \phi_1+\phi_3=\phi_2+\phi_4.$$

Ex. 10. Interpret the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-\frac{zz'}{c^2}\right)^2,$$

where P, (x', y', z') is a point on the hyperboloid.

[The equation represents the pair of planes through the origin and the generators that intersect at (x', y', z').]

Ex. 11. Prove that the generators through any point P on a hyperboloid are parallel to the asymptotes of the section of the hyperboloid by any plane which is parallel to the tangent plane at P.

Ex. 12. Prove that the angle between the generators through P is given by

$$\cot\theta=\frac{p(r^2-a^2-b^2+c^2)}{2abc},$$

where p is the perpendicular from the centre to the tangent plane at P and r is the distance of P from the centre.

Ex. 13. All parallelepipeds which have six of their edges along generators of a given hyperboloid have the same volume.

If PQRS is one face of the parallelepiped and P, P′; Q, Q′; R, R′; S, S′ are opposite corners, we may have the edges PS, RP′, S′R′ along generators of one system and the edges SR, P′S′, R′P along generators of the other system. The tangent planes at S and S′ are therefore PSR, P′S′R′, and are parallel, and therefore SS′ is a diameter. Similarly, PP′ and RR′ are diameters. Let P, S, R be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). Then the volume of the parallelepiped is twelve times the volume of the tetrahedron OPSR, (O is the centre). Denoting it by V, we have

$$\mathrm{V}=2\begin{vmatrix} x & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}=2abc\sqrt{-1}\begin{vmatrix} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{\sqrt{-c^2}} \\ \frac{x_2}{a}, & \frac{y_2}{b}, & \frac{z_2}{\sqrt{-c^2}} \\ \frac{x_3}{a}, & \frac{y_3}{b}, & \frac{z_3}{\sqrt{-c^2}} \end{vmatrix}$$

$$=2abc\sqrt{-1}\begin{vmatrix} \Sigma\frac{x_1^2}{a^2}, & \Sigma\frac{x_1x_2}{a^2}, & \Sigma\frac{x_1x_3}{a^2} \\ \Sigma\frac{x_1x_2}{a^2}, & \Sigma\frac{x_2^2}{a^2}, & \Sigma\frac{x_2x_3}{a^2} \\ \Sigma\frac{x_1x_3}{a^2}, & \Sigma\frac{x_2x_3}{a^2}, & \Sigma\frac{x_3^2}{a^2} \end{vmatrix}^{\frac{1}{2}}.$$

But
$$\Sigma\frac{x_1^2}{a^2}\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1,\quad \Sigma\frac{x_2^2}{a^2}=1,\quad \Sigma\frac{x_3^2}{a^2}=1;$$

and, since R′ and S are on the tangent plane at P, and S on the tangent plane at R,

$$\Sigma\frac{x_1x_2}{a^2}=1,\quad \Sigma\frac{x_2x_3}{a^2}=1,\quad \Sigma\frac{x_1x_3}{a^2}=-1,$$

Therefore
$$\mathrm{V}=2abc\sqrt{-1}(-4)^{\frac{1}{2}}=4abc.$$

Ex. 14. Find the locus of the corners Q and Q′ which are not on the given hyperboloid.

Since QS and PR bisect one another, Q is the point

$$(x_1-x_2+x_3,\quad y_1-y_2+y_3,\quad z_1-z_2+z_3),$$

and hence lies on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+3=0.$$

106. The systems of generators of the hyperbolic paraboloid. We shall now state the results for the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ corresponding to those which we have proved for the hyperboloid. Their proof is left as an exercise for the student.

The point of intersection of the generators

$$\frac{x}{a}-\frac{y}{b}=2\lambda,\quad \frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}; \quad \text{..................(1)}$$

$$\frac{x}{a}+\frac{y}{b}=2\mu,\quad \frac{x}{a}-\frac{y}{b}=\frac{z}{\mu} \quad \text{....................(2)}$$

is given by

$$\frac{x}{a}=\mu+\lambda,\quad \frac{y}{b}=\mu-\lambda,\quad z=2\lambda\mu.$$

The direction-cosines of the generators are given by

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{2\lambda};\quad \frac{l}{a}=\frac{m}{-b}=\frac{n}{2\mu},$$

and hence the locus of the points of intersection of perpendicular generators is the curve of intersection of the surface and the plane $2z+a^2-b^2=0$.

The plane

$$\frac{x}{a}-\frac{y}{b}-2\lambda+k\left\{\frac{x}{a}+\frac{y}{b}-\frac{z}{\lambda}\right\}=0$$

passes through the generator (1) and is tangent plane at the point of intersection of that generator and the generator of the μ-system given by $\mu=\lambda/k$.

The projections of the generators on the planes YOZ, ZOX envelope the principal sections whose equations are

$$x=0,\ y^2=-2b^2z;\quad y=0,\ x^2=2a^2z.$$

Any point on the second parabola is $(2am, 0, 2m^2)$, and if $\lambda=\mu=m$, the generators of the λ- and μ-systems corresponding to these values project into the tangent to the parabola at "m."

Any point on the surface is given by

$$x=ar\cos\theta,\quad y=br\sin\theta,\quad 2z=r^2\cos 2\theta,$$

and the equations to the generators through "r, θ" are

$$\frac{x-ar\cos\theta}{a}=\frac{y-br\sin\theta}{\pm b}=\frac{z-\dfrac{r^2}{2}\cos 2\theta}{r(\cos\theta\mp\sin\theta)}.$$

Ex. 1. Shew that the angle between the generating lines through (x, y, z) is given by

$$\tan\theta=ab\left(1+\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{\frac{1}{2}}\left(z+\frac{a^2-b^2}{2}\right)^{-1}.$$

Ex. 2. Prove that the equations

$$4x=a(1+\cos 2\theta),\quad y=b\cosh\phi\cos\theta,\quad z=c\sinh\phi\cos\theta$$

determine a hyperbolic paraboloid, and that the angle between the generators through "θ, ϕ" is given by

$$\sec\psi=\frac{\{(b^2+c^2)^2+a^4\cos^4\theta+2a^2(b^2+c^2)\cos^2\theta\cosh 2\phi\}^{\frac{1}{2}}}{b^2-c^2+a^2\cos^2\theta}.$$

Ex. 3. Prove that the equations

$$2x=ae^{2\phi},\quad y=be^{\phi}\cosh\theta,\quad z=ce^{\phi}\sinh\theta$$

determine a hyperbolic paraboloid, and that $\theta+\phi$ is constant for points of a given generator of one system, and $\theta-\phi$ is constant for a given generator of the other.

Ex. 4. Planes are drawn through the origin, O, and the generators through any point P of the paraboloid given by $x^2-y^2=az$. Prove that the angle between them is $\tan^{-1}\dfrac{2r}{a}$, where r is the length of OP.

Ex. 5. Find the locus of the perpendiculars from the vertex of the paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$ to the generators of one system.

Ans. $x^2+y^2+2z^2\pm\dfrac{a^2+b^2}{ab}xy=0.$

Ex. 6. The points of intersection of generators of $xy=az$ which are inclined at a constant angle α lie on the curve of intersection of the paraboloid and the hyperboloid $x^2+y^2-z^2\tan^2\alpha+a^2=0$.

107. Conicoids through three given lines. The general equation to a conicoid,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

contains nine constants, viz., the ratios of any nine of the ten coefficients a, b, c, ... to the tenth. Hence, since these are determined by nine equations involving them, a conicoid can be found to pass through nine given points. But we have proved that if three points of a straight line lie on a given conicoid, the line is a generator of the conicoid. Therefore a conicoid can be found to pass through any three given non-intersecting lines.

108. The general equation to a conicoid through the two given lines $u=0=v$, $u'=0=v'$, is

$$\lambda uu'+\mu uv'+\nu vu'+\rho vv'=0,$$

since this equation is satisfied when $u=0$ and $v=0$, or when $u'=0$ and $v'=0$, and contains three disposable constants, viz. the ratios of λ, μ, ν to ρ.

109. *To find the equation to the conicoid through three given non-intersecting lines.*

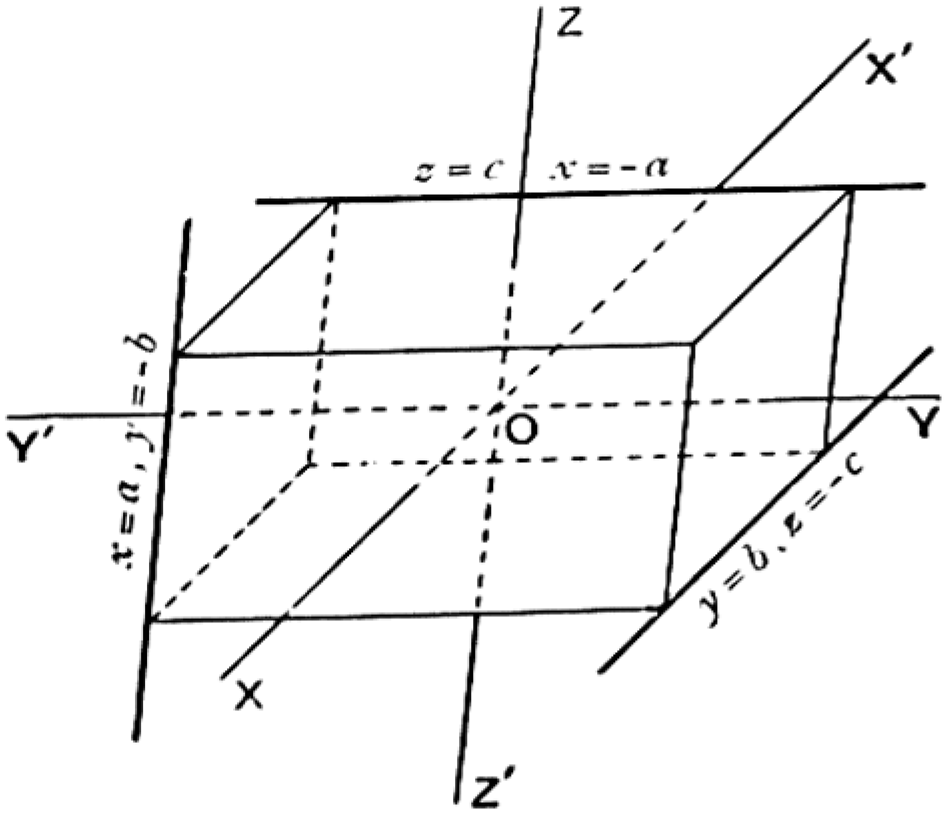

Fig. 45.

If the three lines are not parallel to the same plane, planes drawn through each line parallel to the other two form a parallelepiped, (fig. 45). If the centre of the parallelepiped is taken as origin, and the axes are parallel to the edges, the equations to the given lines are of the form,

$$(1)\ y=b,\ z=-c;\quad (2)\ z=c,\ x=-a;\quad (3)\ x=a,\ y=-b,$$

where $2a$, $2b$, $2c$ are the edges. The general equation to a conicoid through the lines (1) and (2) is

$$(y-b)(z-c)+\lambda(y-b)(x+a)$$
$$+\mu(z+c)(z-c)+\nu(z+c)(x+a)=0.$$

Where $x=a$, $y=-b$ meets the surface we have

$$\mu z^2+2z(a\nu-b)-\mu c^2+2c(a\nu+b)-4ab\lambda=0,$$

and if $x=a$, $y=-b$ is a generator, this equation must be satisfied for *all* values of z. Therefore

$$\mu=0,\quad \nu=\frac{b}{a},\quad \lambda=\frac{c(a\nu+b)}{2ab}=\frac{c}{a},$$

and the equation to the surface is

$$ayz+bzx+cxy+abc=0.$$

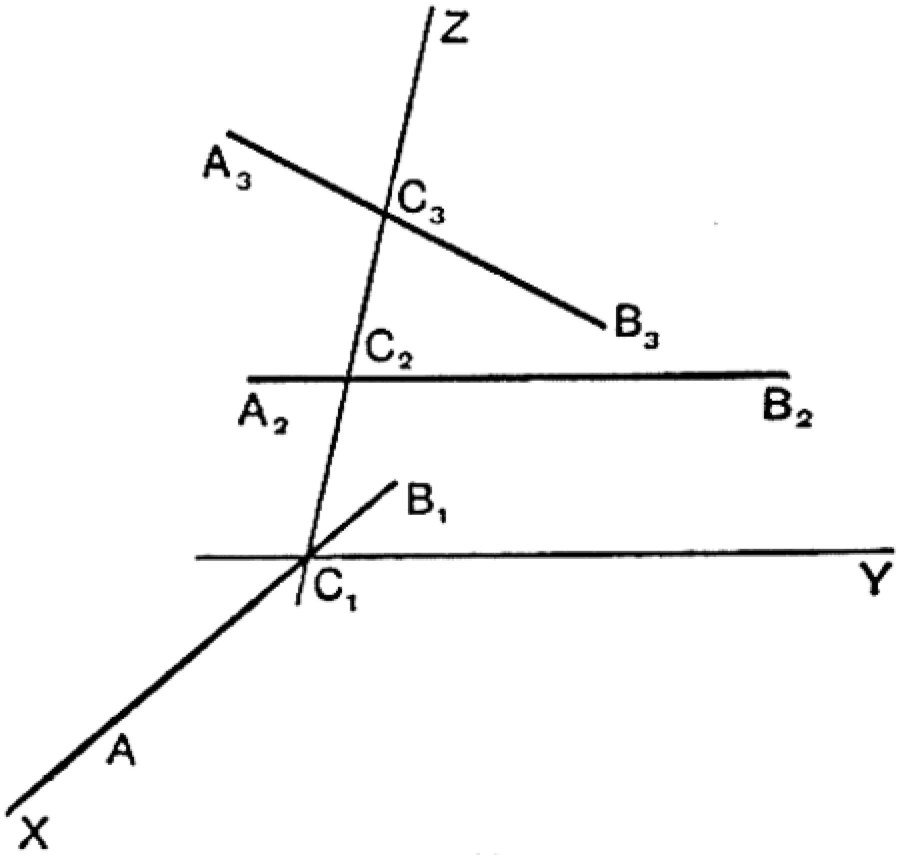

FIG. 46.

The origin evidently bisects all chords of the surface which pass through it, and therefore the surface is a central surface, and is therefore a hyperboloid of one sheet. (Cf. § 47, Ex. 1.)

If the three lines are parallel to the same plane, let any line which meets them be taken as z-axis. If the lines are A_1B_1, A_2B_2, A_3B_3, (fig. 46), and the z-axis meets them in

C_1, C_2, C_3, take A_1B_1 as x-axis and the parallel to A_2B_2 through C_1 as y-axis. Then the equations may be written

$$(1)\ y=0,\ z=0;\quad (2)\ x=0,\ z=\alpha;\quad (3)\ lx+my=0,\ z=\beta.$$

The equation to a conicoid through the lines (2) and (3) is

$$\lambda x(lx+my)+\mu x(z-\beta)$$
$$+\nu(z-\alpha)(lx+my)+\rho(z-\alpha)(z-\beta)=0.$$

If $y=0$, $z=0$ is a generator, the equation

$$l\lambda x^2-x(\mu\beta+\nu l\alpha)+\rho\alpha\beta=0$$

must be satisfied for all values of x, and therefore

$$\lambda=\rho=0,\quad \mu\beta+\nu l\alpha=0;$$

and hence the equation to the surface is

$$z\{lx(\alpha-\beta)-\beta my\}+\alpha\beta my=0.$$

Since the terms of second degree are the product of linear factors, the equation represents a hyperbolic paraboloid.

110. The straight lines which meet four given lines If A, B, C are three given non-intersecting lines, an infinite number of straight lines can be drawn to meet A, B, and C. For a conicoid can be drawn through A, B, C, and A, B, C are generators of one system, say the λ-system, and hence all the generators of the μ-system will intersect A, B, and C.

A fourth line, D, which does not meet A, B, and C, meets the conicoid in general in two points P and Q, and the generators of the μ-system through P and Q are the only lines which intersect the four given lines A, B, C, D. If, however, D is a generator of the conicoid through A, B, and C, it belongs to the λ-system, and therefore all the generators of the μ-system meet all the four lines.

111. *If three straight lines can be drawn to meet four given non-intersecting lines* A, B, C, D, *then* A, B, C, D *are generators of a conicoid.*

If the three lines are P, Q, R, each meets the conicoid through A, B, C in three points, and is therefore a generator. Hence D meets the conicoid in three points, viz. the points of intersection of D and P, Q, R; and therefore D is a generator.

Ex. 1. **A, A′; B, B′; C, C′** are points on **X′OX, Y′OY, Z′OZ.** Prove that **BC′, CA′, AB′** are generators of one system, and that **B′C, C′A, A′B** are generators of the other system, of a hyperboloid.

Ex. 2. **A, A′; B, B′; C, C′** are pairs of opposite vertices of a skew hexagon drawn on a hyperboloid. Prove that **AA′, BB′, CC′** are concurrent.

Ex. 3. The altitudes of a tetrahedron are generators of a hyperboloid of one sheet.

Let **A, B, C, D** be the vertices. Then the planes through **DA**, perpendicular to the plane **DBC**, through **DB**, perpendicular to the plane **DCA**, and through **DC**, perpendicular to the plane **DAB**, pass through one line, (§ 45, Ex. 6, or § 44, Ex. 22). That line is therefore coplanar with the altitudes from **A, B, C**, and it meets the altitude from **D** in **D**, and therefore it meets all the four altitudes. The corresponding lines through **A, B, C** also meet all the four altitudes, which are therefore generators of a hyperboloid.

Ex. 4. Prove that the perpendiculars to the faces of the tetrahedron through their orthocentres are generators of the opposite system.

Ex. 5. Prove that the lines joining **A, B, C, D** to the centres of the circles inscribed in the triangles **BCD, CDA, DAB, ABC** are generators of a hyperboloid.

112. The equation to a hyperboloid when two intersecting generators are coordinate axes. If two intersecting generators are taken as x-axis and y-axis, the equation to the surface must be satisfied by all values of x when $y=z=0$, and by all values of y when $z=x=0$.

Suppose that it is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz=0.$$

Then we must have

$$a=u=0, \text{ and } b=v=0,$$

and therefore the equation takes the form

$$cz^2+2fyz+2gzx+2hxy+2wz=0.$$

Suppose now that the line joining the point of intersection of the generators to the centre is taken as z-axis. Then, since the generators through opposite ends of a diameter are parallel, the lines $y=0$, $z=2\gamma$; $x=0$, $z=2\gamma$ are generators, the centre being $(0, 0, \gamma)$. Whence

$$f=g=0, \quad \gamma=-w/c,$$

and the equation reduces to

$$cz^2+2hxy+2wz=0.$$

Ex. 1. Prove that $(y+mz)(z+nx)+kz=0$ represents a paraboloid which passes through OX and OY.

Ex. 2. The generators through a variable point P of a hyperboloid meet the generators through a fixed point O in Q and R. If OQ : OR is constant, find the locus of P.

Take OQ and OR as x- and y-axes, and the line joining O to the centre as z-axis. The equation to the hyperboloid is

$$cz^2+2hxy+2wz=0.$$

It may be written $\quad z(cz+2w)+2hxy=0,$

and hence the systems of generating lines are given by

$$z=2h\lambda x, \quad \lambda(cz+2w)+y=0;$$
$$z=2h\mu y, \quad \mu(cz+2w)+x=0.$$

OX belongs to the λ-system and corresponds to $\lambda=0$; OY belongs to the μ-system and corresponds to $\mu=0$. If P is (ξ, η, ζ), the generators through P correspond to

$$\lambda=\zeta/2h\xi, \quad \mu=\zeta/2h\eta.$$

Where a generator of the μ-system meets OX,

$$y=0, \quad z=0, \quad x=-2w\mu,$$

therefore $\quad \text{OQ}=-2w\mu=-w\zeta/h\eta.$

Similarly, $\quad \text{OR}=-2w\lambda=-w\zeta/h\xi,$

and P therefore lies on the plane $x=ky$.

[OQ and OR may be found more easily by considering that the plane PQR is the tangent plane at P whose equation (see § 134) is $h\eta x+h\xi y+(c\zeta+w)z+w\zeta=0$.]

Ex. 3. Find the locus of P if (i) $\text{OQ}\cdot\text{OR}=k^2$, (ii) $\text{OQ}^2+\text{OR}^2=k^2$.

Ex. 4. If $\text{OQ}^{-2}+\text{OR}^{-2}$ is constant, P lies on a cone whose vertex is O and whose section by a plane parallel to OXY is an ellipse whose equal conjugate diameters are parallel to OX and OY.

Ex. 5. Shew that the projections of the generators of one system of a hyperboloid on the tangent plane at any point envelope a conic.

Take the generators in the given tangent plane as OX and OY, and the normal at O as OZ. The plane $z=\lambda y$ is a tangent plane, (§ 97), and the projection on OXY of the second generator in which it meets the surface has equations

$$z=0, \quad c\lambda^2 y+2\lambda(gx+fy+w)+2hx=0.$$

Whence the envelope of the projections is the conic

$$z=0, \quad (gx+fy+w)^2=2chxy.$$

*113. Properties of a given generating line.

If we have a system of rectangular axes in which the x-axis is a generator and the z-axis is the normal at the origin, the equation to the hyperboloid is of the form,

$$by^2+cz^2+2fyz+2gzx+2hxy+2wz=0,$$

or $\quad y(by+2hx)+z(cz+2gx+2fy+2w)=0.$

The systems of generating lines are given by

$$\lambda y = z, \quad (by + 2hx) + \lambda(cz + 2gx + 2fy + 2w) = 0;$$
$$y = \mu(cz + 2gx + 2fy + 2w), \quad z + \mu(by + 2hx) = 0.$$

The x-axis belongs to the μ-system and corresponds to $\mu = 0$. The generator of the λ-system through the point $(\alpha, 0, 0)$ is given by

$$\lambda = \frac{-h\alpha}{g\alpha + w}.$$

The tangent plane at $(\alpha, 0, 0)$ is the plane through this generator and **OX**. Its equation is therefore $\lambda y = z$, or

$$h\alpha y + z(g\alpha + w) = 0.$$

Let P, $(\alpha, 0, 0)$, P′, $(\alpha', 0, 0)$ be points on the x-axis. Then the tangent planes at P and P′ are at right angles if

$$(h^2 + g^2)\alpha\alpha' + wg(\alpha + \alpha') + w^2 = 0,$$

$$i.e. \text{ if } \left(\alpha + \frac{wg}{h^2 + g^2}\right)\left(\alpha' + \frac{wg}{h^2 + g^2}\right) = \frac{-w^2h^2}{(h^2 + g^2)^2}. \quad \ldots\ldots(1)$$

Therefore if C is the point $\left(-\frac{wg}{h^2 + g^2}, 0, 0\right)$, CP · CP′ is constant for all pairs of perpendicular tangent planes through OX. C is called the **central point** of the generator OX. If the origin is taken at the central point, the equation (1) must take the form $\alpha\alpha' = \text{constant}$, and therefore $g = 0$, and $\alpha\alpha' = -w^2/h^2$. The equation to the conicoid when OX is a generator and O is the central point, OZ is the normal at O, and the axes are rectangular, is therefore

$$by^2 + cz^2 + 2fyz + 2hxy + 2wz = 0.$$

Ex. 1. Find the locus of the normals to a conicoid at points of a given generator.

Taking axes as above, the equations to the normal at $(\alpha, 0, 0)$ are $\frac{x - \alpha}{0} = \frac{y}{h\alpha} = \frac{z}{w}$. The locus of the normals is therefore the hyperbolic paraboloid whose equation is $hxz = wy$. It has OX and OZ as generators, and its vertex at the origin.

Ex. 2. The anharmonic ratio of four tangent planes through the same generator is the anharmonic ratio of their points of contact.

The tangent plane at $(\alpha_r, 0, 0)$ is $h\alpha_r y + wz = 0$, whence, by § 38, Ex. 4, the anharmonic ratio of the planes is

$$\frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_3 - \alpha_2)(\alpha_1 - \alpha_4)}.$$

Ex. 3. Four fixed generators of the same system meet any generator of the opposite system in a range of constant anharmonic ratio.

Ex. 4. Find the locus of the perpendiculars from a point on a hyperboloid to the generators of one system.

Take O, the point, as origin, and a generator through O as OX. Take the normal at O as OZ, then XOY is the tangent plane at O. The equation to the hyperboloid is

$$by^2+cz^2+2fyz+2gzx+2hxy+2wz=0.$$

The systems of generators are given by

$$\lambda y=z, \quad (by+2hx)+\lambda(cz+2gx+2fy+2w)=0\ ;$$

$$y=\mu(cz+2gx+2fy+2w), \quad z+\mu(by+2hx)=0.$$

The locus of the perpendiculars to the generators of the λ-system is the cubic cone

$$x(cz^2+2fyz+by^2)-2(y^2+z^2)(hy+gz)=0.$$

*114. The central point and parameter of distribution.

Taking the axes indicated in § 113 the equation to the conicoid is

$$by^2+cz^2+2fyz+2hxy+2wz=0.$$

The equations to the system of generators to which OX belongs are

$$y=\lambda(2fy+cz+2w), \quad z+\lambda(2hx+by)=0,$$

OX being given by $\lambda=0$. The direction-cosines of a generator of this system are proportional to

$$bc\lambda^2-2f\lambda+1, \quad -2ch\lambda^2, \quad 2h\lambda(2f\lambda-1),$$

and therefore the shortest distance between this generator and OX has direction-cosines proportional to

$$0, \quad 2f\lambda-1, \quad c\lambda.$$

Hence the limiting position of the shortest distance, as λ tends to zero, is parallel to OY. Again, any plane through the generator is given by

$$y(2f\lambda-1)+c\lambda z+2w\lambda-k\{2h\lambda x+b\lambda y+z\}=0.$$

This plane meets OX where $x=w/hk$. It contains the S.D. if

$$(2f\lambda-1)(2f\lambda-1-bk\lambda)+c\lambda(c\lambda-k)=0,$$

i.e. if

$$k=\frac{1-4f\lambda}{(c-b)\lambda},$$

squares and higher powers of λ being rejected.

Therefore the S.D. meets **OX** where

$$x=\frac{w(c-b)\lambda}{h(1-4f\lambda)}.$$

Since x tends to zero with λ, the limiting position of the S.D. is **OY**. Hence the central point of a given generator is the point of intersection of the generator and the shortest distance between it and a consecutive generator of the same system.

The equation $y=\lambda(2fy+cz+2w)$ represents the plane through the λ-generator parallel to **OX**. Therefore the shortest distance, δ, is given by

$$\delta=\frac{2w\lambda}{\sqrt{(1-2f\lambda)^2+c^2\lambda^2}}=2w\lambda,$$

rejecting λ^2, etc.

Again, if θ is the angle between the generator and **OX**,

$$\cos\theta=\frac{bc\lambda^2-2f\lambda+1}{\sqrt{(bc\lambda^2-2f\lambda+1)^2+4c^2h^2\lambda^4+4h^2\lambda^2(2f\lambda-1)^2}},$$

whence, if λ^2 and higher powers be rejected,

$$\theta=2h\lambda.$$

The limit of the ratio δ/θ, as λ tends to zero, is called the **parameter of distribution** of the generator **OX**. Denoting it by p, we have

$$p=\mathrm{Lt}\,\frac{2w\lambda}{2h\lambda}=\frac{w}{h}.$$

Cor. If **O** is the central point and the tangent planes at **A** and **A′** are at right angles, **OA . OA′** $=-p^2$.

Ex. 1. If the generator "ϕ" of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is given by

$$\frac{x-a\cos\phi}{a\sin\phi}=\frac{y-b\sin\phi}{-b\cos\phi}=\frac{z}{-c},$$

and θ is the angle between the generators "ϕ" and "ϕ_1," prove that

$$\sin^2\theta=\frac{a^2b^2\sin^2(\phi-\phi_1)+a^2c^2(\sin\phi-\sin\phi_1)^2+b^2c^2(\cos\phi-\cos\phi_1)^2}{(a^2\sin^2\phi+b^2\cos^2\phi+c^2)(a^2\sin^2\phi_1+b^2\cos^2\phi_1+c^2)},$$

and deduce that

$$\frac{d\theta}{d\phi}=\frac{(a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi)^{\frac{1}{2}}}{a^2\sin^2\phi+b^2\cos^2\phi+c^2}.$$

Ex. 2. Prove that the shortest distance, δ, between the generators "ϕ" and "ϕ_1" is given by

$$\delta=\frac{2abc\sin\dfrac{\phi_1-\phi}{2}}{\left(b^2c^2\sin^2\dfrac{\phi_1+\phi}{2}+c^2a^2\cos^2\dfrac{\phi_1+\phi}{2}+a^2b^2\cos^2\dfrac{\phi_1-\phi}{2}\right)^{\frac{1}{2}}}$$

and deduce that $\dfrac{d\delta}{d\phi}=\dfrac{abc}{(a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi)^{\frac{1}{2}}}$.

Ex. 3. Prove that the parameter of distribution for the generator "ϕ" is

$$\frac{abc(a^2\sin^2\phi+b^2\cos^2\phi+c^2)}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi}.$$

Ex. 4. If D is the distance of any generator of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

from the centre, and p is its parameter of distribution, $D^2p=abc$.

Ex. 5. Find the coordinates of the central point of the generator "ϕ."

The equation to the plane through the generator "ϕ" parallel to the generator "ϕ_1" is

$$\frac{x}{a}\sin\frac{\phi+\phi_1}{2}-\frac{y}{b}\cos\frac{\phi+\phi_1}{2}+\frac{z}{c}\cos\frac{\phi-\phi_1}{2}+\sin\frac{\phi-\phi_1}{2}=0.$$

Whence the direction-cosines of the S.D. between the generator "ϕ" and a consecutive generator of the same system are proportional to

$$\frac{1}{a}\sin\phi,\quad -\frac{1}{b}\cos\phi,\quad \frac{1}{c}.$$

The coordinates of any point, O, on the generator are

$$a(\cos\phi-k\sin\phi),\quad b(\sin\phi+k\cos\phi),\quad ck.$$

If O is the central point the normal at O is perpendicular to the S.D. between the generator and a consecutive generator of the same system. Hence we find

$$k=\frac{c^2(b^2-a^2)\sin\phi\cos\phi}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi},$$

and the coordinates of the central point are given by

$$\frac{x}{a^3(b^2+c^2)\cos\phi}=\frac{y}{b^3(c^2+a^2)\sin\phi}=\frac{z}{c^3(b^2-a^2)\sin\phi\cos\phi}$$
$$=\frac{1}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi}.$$

Ex. 6. Find the locus of the central points of the generators of the hyperboloid.

The equation to a surface containing the central points is obtained by eliminating ϕ between the equations for the coordinates. It is

$$\frac{a^6}{x^2}(b^2+c^2)^2+\frac{b^6}{y^2}(c^2+a^2)^2-\frac{c^6}{z^2}(b^2-a^2)^2=0.$$

Ex. 7. For the generator of the paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ given by $\frac{x}{a}-\frac{y}{b}=2\lambda$, $\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}$, prove that the parameter of distribution is $\frac{ab(a^2+b^2+4\lambda^2)}{a^2+b^2}$, and that the central point is

$$\left(\frac{2a^3\lambda}{a^2+b^2}, \frac{-2b^3\lambda}{a^2+b^2}, \frac{2(a^2-b^2)\lambda^2}{a^2+b^2}\right).$$

Prove also that the central points of the systems of generators lie on the planes $\frac{x}{a^3}\pm\frac{y}{b^3}=0$.

Ex. 8. If G is a given generator of a hyperboloid, prove that the tangent plane at the central point of G is perpendicular to the tangent plane to the asymptotic cone whose generator of contact is parallel to G.

Ex. 9. A pair of planes through a given generator of a hyperboloid touch the surface at points A and B, and contain the normals at points A′ and B′ of the generator. If θ is the angle between them, prove that $\tan^2\theta=-\frac{AB\,.\,A'B'}{AB'\,.\,A'B}$.

Ex. 10. If the tangent plane at a point P of a generator, central point O, makes an angle θ with the tangent plane at O, $p\tan\theta=OP$, where p is the parameter of distribution.

*Examples VI.

1. Prove that the line $lx+my+nz+p=0$, $l'x+m'y+n'z+p'=0$ is a generator of the hyperboloid $x^2/a+y^2/b+z^2/c=1$ if $al^2+bm^2+cn^2=p^2$, $al'^2+bm'^2+cn'^2=p'^2$, and $all'+bmm'+cnn'=pp'$.

2. Shew that the equations

$$y-\lambda z+\lambda+1=0, \quad (\lambda+1)x+y+\lambda=0$$

represent for different values of λ generators of one system of the hyperboloid $yz+zx+xy+1=0$, and find the equations to generators of the other system.

3. Tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, which are parallel to tangent planes to

$$\frac{b^2c^2x^2}{c^2-b^2}+\frac{c^2a^2y^2}{c^2-a^2}+\frac{a^2b^2z^2}{a^2+b^2}=0,$$

cut the surface in perpendicular generators.

4. The shortest distances between generators of the same system drawn at the ends of diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on the surfaces whose equations are

$$\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}.$$

5. Shew that the shortest distance of any two perpendicular members of that system of generators of the paraboloid $y(ax+by)=z$, which is perpendicular to the y-axis, lies in the plane $a^2z=b$.

6. Prove that any point on the lines

$$x+1=\mu y=-(\mu+1)z$$

lies on the surface $\quad yz+zx+xy+y+z=0,$

and find equations to determine the other system of lines which lies on the surface.

7. The four conicoids, each of which passes through three of four given non-intersecting lines, have two common generators.

8. Prove that the equation to the conicoid through the lines

$$u=0=v, \quad u'=0=v',$$

$$\lambda u+\mu v+\lambda' u'+\mu' v'=0=lu+mv+l'u'+m'v'$$

is

$$\frac{\lambda u+\mu v}{\lambda' u'+\mu' v'}=\frac{lu+mv}{l'u'+m'v'}.$$

9. Show that, in general, two generators of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

can be drawn to cut a given generator at right angles, and that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse

$$x^2/a^6+y^2/b^6=c^4/a^4b^4.$$

10. If from a fixed point on a hyperboloid lines are drawn to intersect the diagonals of the quadrilaterals formed by two fixed and two variable generators, these lines are coplanar.

11. Through a variable generator $x-y=\lambda$, $x+y=2z/\lambda$ of the paraboloid $x^2-y^2=2z$ a plane is drawn making a constant angle α with the plane $x=y$. Find the locus of the point at which it touches the paraboloid.

12. Prove that the locus of the line of intersection of two perpendicular planes which pass through two fixed non-intersecting lines is a hyperboloid whose central circular sections are perpendicular to the lines and have their diameters equal to their shortest distance.

13. Prove that if the generators of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ be drawn through the points where it is met by a tangent to

$$z=0, \quad \frac{x^2}{a^2(a^2+c^2)}+\frac{y^2}{b^2(b^2+c^2)}=\frac{1}{a^2+b^2},$$

they form a skew quadrilateral with two opposite angles right angles, and the other diagonal of which is a generator of the cylinder

$$\frac{x^2(a^2+c^2)}{a^2}+\frac{y^2(b^2+c^2)}{b^2}=a^2+b^2.$$

14. The normals to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ at points of a generator meet the plane $z=0$ at points lying on a straight line, and for different generators of the same system this line touches a fixed conic.

15. Prove that the generators of $ax^2+by^2+cz^2=1$ through (x_1, y_1, z_1), (x_2, y_2, z_2) lie in the planes

$$(ax_1x_2+by_1y_2+cz_1z_2-1)(ax^2+by^2+cz^2-1)$$
$$=2(axx_1+byy_1+czz_1-1)(axx_2+byy_2+czz_2-1).$$

16. The generators through points on the principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, such that the eccentric angle of the one is double the eccentric angle of the other, intersect on the curves given by

$$x=\frac{a(1-3t^2)}{1+t^2},\quad y=\frac{bt(3-t^2)}{1+t^2},\quad z=\pm ct.$$

17. The planes of triangles which have a fixed centre of gravity and have their vertices on three given straight lines which are parallel to the same plane, touch a cone of the second degree, and their sides are generators of three paraboloids.

18. The cubic curve

$$x=\frac{1}{\lambda-\alpha},\quad y=\frac{1}{\lambda-\beta},\quad z=\frac{1}{\lambda-\gamma}$$

meets the conicoid $ax^2+by^2+cz^2=1$ in six points, and the normals at these points are generators of the hyperboloid

$$ayz(\beta-\gamma)+bzx(\gamma-\alpha)+cxy(\alpha-\beta)+x(b-c)+y(c-a)+z(a-b)=0.$$

19. Prove that the locus of a point whose distances from two given lines are in a constant ratio is a hyperboloid of one sheet, and that the projections of the lines on the tangent plane at the point where it meets the shortest distance form a harmonic pencil with the generators through the point.

20. The generators through **P** on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meet the plane $z=0$ in **A** and **B**. If **PA** : **PB** is constant, find the locus of **P**.

21. If the median of the triangle **PAB** in the last example is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, shew that **P** lies on the surface

$$z(\alpha x+\beta y)+\gamma(c^2+z^2)=0.$$

22. If **A** and **B** are the extremities of conjugate diameters of the principal elliptic section, prove that the median through **P** of the triangle **PAB** lies on the cone

$$\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=\left(\frac{z}{c}\pm 1\right)^2.$$

23. **A** and **B** are the extremities of the axes of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, and **T** is any line in the plane of the section. **G**$_1$, **G**$_2$ are generators of the same system, **G**$_1$ passing through **A** and **G**$_2$ through **B**. Two hyperboloids are drawn, one through **T**, **G**$_1$, **OZ**, the other through **T**, **G**$_2$, **OZ**. Shew that the other common generators of these hyperboloids lie on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z}{c}\left(\frac{x}{a}-\frac{y}{b}\right)-\frac{x}{a}-\frac{y}{b}=0.$$

24. Prove that the shortest distances between the generator

$$\frac{x}{a}=\frac{z}{c}\cos\alpha-\sin\alpha,\quad \frac{y}{b}=\frac{z}{c}\sin\alpha+\cos\alpha,$$

and the other generators of the same system, meet the generators in points lying in the plane

$$\frac{x\cos\alpha}{a}(a^{-2}-b^{-2}+c^{-2})+\frac{y\sin\alpha}{b}(-a^{-2}+b^{-2}+c^{-2})+\frac{z}{c}(a^{-2}+b^{-2}+c^{-2})=0.$$

25. If the generators through **P**, a point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

whose centre is **O**, meet the plane $z=0$ in **A** and **B**, and the volume of the tetrahedron **OAPB** is constant and equal to $abc/6$, **P** lies on one of the planes $z=\pm c$.

APPENDIX

Note to § 37 : Distance from a point to a plane.

When the axes are rectangular, the distance from the point P, (x', y', z') to the plane $ax+by+cz+d=0$ can be found as follows.

Let N, (α, β, γ) be the foot of the perpendicular from P to the plane. Then $a\alpha+b\beta+c\gamma+d=0$, and the direction of NP is given by

$$x'-\alpha : y'-\beta : z'-\gamma,$$

or, since NP is parallel to the normal, by $a : b : c$.

Therefore
$$\frac{x'-\alpha}{a}=\frac{y'-\beta}{b}=\frac{z'-\gamma}{c}=\frac{ax'+by'+cz'-(a\alpha+b\beta+c\gamma)}{a^2+b^2+c^2},$$
$$=\frac{ax'+by'+cz'+d}{a^2+b^2+c^2}=u, \text{ say.}$$

$$\therefore\ x'-\alpha=au,\ y'-\beta=bu,\ z'-\gamma=cu, \quad \dots\dots(1)$$

and
$$\mathrm{NP}^2=(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2,$$
$$=(a^2+b^2+c^2)u^2.$$

Hence
$$\mathrm{NP}=\frac{ax'+by'+cz'+d}{\pm\sqrt{a^2+b^2+c^2}}.$$

From (1), N is given by $\alpha=x'-au$, $\beta=y'-bu$, $\gamma=z'-cu$.

Note to § 43 : Constants in the equations of a line.

The constants may be put in a more symmetrical form as follows :

If $nb-mc=\lambda$, $lc-na=\mu$, $ma-lb=\nu$, so that $l\lambda+m\mu+n\nu=0$, the equations $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$ are equivalent to

$$ny-mz=\lambda,\quad lz-nx=\mu,\quad mx-ly=\nu, \quad \dots\dots(1)$$

from which
$$\lambda x+\mu y+\nu z=0, \quad \dots\dots(2)$$

The equations (1) represent the planes through the line parallel to the coordinate axes ; the equation (2) represents the plane through the line and the origin.

Since $l\lambda+m\mu+n\nu=0$, the ratios of any five of the quantities l, m, n, λ, μ, ν, to the remaining one are equivalent to four independent constants, and, when they are given, the line is determined. If, for example, the ratios of m, n, λ, μ, ν to l are given, the equations

$$lz - nx = \mu, \quad mx - ly = \nu,$$

determine fixed planes ; if the ratios of l, m, n, μ, ν to λ are given, the equations

$$ny - mz = \lambda, \quad \lambda x + \mu y + \nu z = 0,$$

determine fixed planes.

The ratios $l : m : n : \lambda : \mu : \nu$ have been called the *coordinates of the line*, and the line is referred to as " the line $(l, m, n, \lambda, \mu, \nu)$." (See Miscellaneous Examples I, Ex. 116.)

Note to § 54 : Section of a surface by a given plane.

The scheme on p. 73 can be written down mechanically. Choosing $O\xi$ and $O\zeta$ as stated, we have

	x	y	z
ξ	$\dfrac{-m}{\sqrt{l^2+m^2}}$	$\dfrac{l}{\sqrt{l^2+m^2}}$	0
η	(1)	(2)	(3)
ζ	l	m	n

where the spaces (1), (2), (3) are to be filled. Now in the determinant

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

of § 53, provided that the orientations of the new and the original axes are the same, each element is equal to its cofactor. Hence in the spaces (1), (2), (3) we may place

$$-\begin{vmatrix} \dfrac{l}{\sqrt{l^2+m^2}}, & 0 \\ m, & n \end{vmatrix}, \quad \begin{vmatrix} \dfrac{-m}{\sqrt{l^2+m^2}}, & 0 \\ l, & n \end{vmatrix}, \quad -\begin{vmatrix} \dfrac{-m}{\sqrt{l^2+m^2}}, & \dfrac{l}{\sqrt{l^2+m^2}} \\ l, & m \end{vmatrix},$$

respectively, or

$$\frac{-ln}{\sqrt{l^2+m^2}}, \quad \frac{-mn}{\sqrt{l^2+m^2}}, \quad \sqrt{l^2+m^2}.$$

Note to § 63 : Cone with a given curve as base.

Let V, (α, β, γ) be the given vertex, and $f(x, y) = 0$, $z = 0$ represent the given base. Let P, (x', y', z') be any point on the cone and let VP cut the base in Q. Then VQ/QP $= -\gamma/z'$, and hence Q is the point

$$\left(\frac{\alpha z' - \gamma x'}{z' - \gamma}, \ \frac{\beta z' - \gamma y'}{z' - \gamma}, \ 0\right).$$

Therefore, since Q lies on the surface $f(x, y)=0$,

$$f\left(\frac{\alpha z'-\gamma x'}{z'-\gamma},\ \frac{\beta z'-\gamma y'}{z'-\gamma}\right)=0,$$

or the coordinates of P satisfy the equation

$$f\left(\frac{\alpha z-\gamma x}{z-\gamma},\ \frac{\beta z-\gamma y}{z-\gamma}\right)=0,$$

which represents the cone.

This equation can be written

$$f\left(\alpha-\frac{\gamma(x-\alpha)}{z-\gamma},\ \beta-\frac{(y-\beta)}{z-\gamma}\right)=0,$$

which, in the case where $f(x, y)$ is of the second degree, can be expanded in the form

$$f(\alpha, \beta)-\gamma\left(\frac{x-\alpha}{z-\gamma}\frac{\partial f}{\partial\alpha}+\frac{y-\beta}{z-\gamma}\frac{\partial f}{\partial\beta}\right)+\gamma^2\phi\left(\frac{x-\alpha}{z-\gamma},\ \frac{y-\beta}{z-\gamma}\right)=0,$$

or $$(z-\gamma)^2 f(\alpha, \beta)-\gamma(z-\gamma)\left(\overline{x-\alpha}\frac{\partial f}{\partial\alpha}+\overline{y-\beta}\frac{\partial f}{\partial\beta}\right)+\gamma^2\phi(x-\alpha, y-\beta)=0,$$

which can then be transformed as in § 63.

A more general case is that of the cone with vertex V, (α, β, γ) and base the curve in which the plane

$$u\equiv lx+my+nz+p=0$$

cuts the surface $f(x, y, z)=0$.

If P, (x', y', z') is any point on the cone, if VP meets the plane in Q, and if VQ/QP $=\lambda/1$, Q is the point

$$\left(\frac{\lambda x'+\alpha}{\lambda+1},\ \frac{\lambda y'+\beta}{\lambda+1},\ \frac{\lambda z'+\gamma}{\lambda+1}\right).$$

Since Q lies on the plane and on the surface,

$$\lambda(lx'+my'+nz'+p)+(l\alpha+m\beta+n\gamma+p)=0, \quad \text{............(1)}$$

and $$f\left(\frac{\lambda x'+\alpha}{\lambda+1},\ \frac{\lambda y'+\beta}{\lambda+1},\ \frac{\lambda z'+\gamma}{\lambda+1}\right)=0 \quad \text{..................(2)}$$

Whence, if $u'\equiv lx'+my'+nz'+p$ and $v\equiv l\alpha+m\beta+n\gamma+p$, (1) and (2) give

$$-\frac{v}{u'}=\lambda, \quad \text{and} \quad f\left(\frac{\alpha u'-vx'}{u'-v},\ \frac{\beta u'-vy'}{u'-v},\ \frac{\gamma u'-vz'}{u'-v}\right)=0,$$

which is the condition that the coordinates of P should satisfy the equation

$$f\left(\frac{\alpha u-vx}{u-v},\ \frac{\beta u-vy}{u-v},\ \frac{\gamma u-vz}{u-v}\right)=0,$$

and this equation represents the cone.

Note to § 96: The generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

The equation of the hyperboloid can also be written

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{x^2}{a^2},$$

whence we deduce that the lines whose equations are

$$\frac{y}{b}+\frac{z}{c}=k\left(1+\frac{x}{a}\right), \quad k\left(\frac{y}{b}-\frac{z}{c}\right)=1-\frac{x}{a}; \quad \ldots\ldots\ldots\ldots\ldots\text{(3)}$$

$$\frac{y}{b}-\frac{z}{c}=k'\left(1+\frac{x}{a}\right), \quad k'\left(\frac{y}{b}+\frac{z}{c}\right)=1-\frac{x}{a}; \quad \ldots\ldots\ldots\ldots\ldots\text{(4)}$$

are generators of the hyperboloid. Adding and subtracting the equations (3), we obtain

$$(1-k)\left(\frac{x}{a}+\frac{z}{c}\right)=(1+k)\left(1-\frac{y}{b}\right), \quad (1+k)\left(\frac{x}{a}-\frac{z}{c}\right)=(1-k)\left(1+\frac{y}{b}\right). \ldots\text{(5)}$$

If now we put μ for $\frac{1-k}{1+k}$ the equations (5) become identical with the equations (2) of § 96. Similarly from (4), by putting λ for $\frac{1-k'}{1+k'}$, we obtain the equations (1) of § 96. Thus different methods of factorising the equation of the hyperboloid lead to the same two systems of generating lines.

Note to § 104: The generators of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ through the point $(a \cos \alpha, b \sin \alpha, 0)$.

The equation of the hyperboloid can be written

$$\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2+\left(\frac{x}{a}\sin\alpha-\frac{y}{b}\cos\alpha\right)^2-\frac{z^2}{c^2}=1.$$

Hence the tangent plane at $(a\cos\alpha, b\sin\alpha, 0)$, whose equation is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$, or $\cos\alpha\left(\frac{x}{a}-\cos\alpha\right)+\sin\alpha\left(\frac{y}{b}-\sin\alpha\right)=0$, cuts the surface where it cuts the planes $\frac{x}{a}\sin\alpha-\frac{y}{b}\cos\alpha=\pm\frac{z}{c}$, that is, in the lines

$$\frac{x/a-\cos\alpha}{\sin\alpha}=\frac{y/b-\sin\alpha}{-\cos\alpha}=\frac{\pm z/c}{\sin^2\alpha+\cos^2\alpha},$$

or

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}.$$

The reduction of the general equation of the second degree (axes rectangular).

The nature of the surfaces represented by the equations

$$\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2=\mu, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

$$\lambda_1\xi^2+\lambda_2\eta^2=2\mu\zeta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}$$

is shown in the following tables :

$$\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2=\mu$$

λ_1	λ_2	λ_3	μ	Surface
+	+	+	+	Ellipsoid (or sphere)
+	+	−	+	Hyperboloid of one sheet
+	−	−	+	Hyperboloid of two sheets
+	+	−	0	Cone
+	+	0	+	Elliptic cylinder
+	−	0	+	Hyperbolic cylinder
+	−	0	0	Pair of intersecting planes
+	0	0	+	Pair of parallel planes

$$\lambda_1\xi^2+\lambda_2\eta^2=2\mu\zeta$$

λ_1	λ_2	μ	Surface
+	+	+	Elliptic paraboloid
+	−	+	Hyperbolic paraboloid
+	0	±	Parabolic cylinder

If we change the origin and rotate the axes, the transformed equation is obtained by substituting for ξ, η, ζ expressions of the form

$$lx+my+nz+p,$$

and therefore it is of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0. \quad \text{......(3)}$$

We shall now indicate how the values of λ_1, λ_2, λ_3, μ in equations (1) and (2) may be found when equation (3) is given.

First we shall prove that if, by a rotation of rectangular axes,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

transforms into

$$\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2,$$

then λ_1, λ_2, λ_3 are the roots of the equation

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0,$$

or

$$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-f^2-g^2-h^2) \\ -(abc+2fgh-af^2-bg^2-ch^2)=0.$$

This equation is called the *Discriminating Cubic*, and it can also be written in the form

$$\lambda^3-\lambda^2(a+b+c)+\lambda(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{D}=0,$$

where $\mathrm{D}\equiv\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, and A, B, C are the cofactors of a, b, c respectively, in D.

Suppose that the scheme for change of axes is

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

, then

$$\begin{aligned}\xi&=l_1x+m_1y+n_1z,\\ \eta&=l_2x+m_2y+n_2z,\\ \zeta&=l_3x+m_3y+n_3z,\end{aligned}$$

and

$$\lambda_1(l_1x+m_1y+n_1z)^2+\lambda_2(l_2x+m_2y+n_2z)^2+\lambda_3(l_3x+m_3y+n_3z)^2$$
$$\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

Whence

$$\begin{aligned} a&=\lambda_1l_1^2+\lambda_2l_2^2+\lambda_3l_3^2, & f&=\lambda_1m_1n_1+\lambda_2m_2n_2+\lambda_3m_3n_3,\\ b&=\lambda_1m_1^2+\lambda_2m_2^2+\lambda_3m_3^2, & g&=\lambda_1n_1l_1+\lambda_2n_2l_2+\lambda_3n_3l_3,\\ c&=\lambda_1n_1^2+\lambda_2n_2^2+\lambda_3n_3^2, & h&=\lambda_1l_1m_1+\lambda_2l_2m_2+\lambda_3l_3m_3.\end{aligned}$$

$$\therefore\ a+b+c=\lambda_1+\lambda_2+\lambda_3,\quad \text{since } \Sigma l_1^2=1, \text{ etc.}$$

Again,
$$\begin{aligned} bc-f^2&=\lambda_2\lambda_3(m_2n_3-m_3n_2)^2+\lambda_3\lambda_1(m_3n_1-m_1n_3)^2\\ &\qquad+\lambda_1\lambda_2(m_1n_2-m_2n_1)^2,\\ &=\lambda_2\lambda_3l_1^2+\lambda_3\lambda_1l_2^2+\lambda_1\lambda_2l_3^2, \quad \text{by § 53 (E)};\end{aligned}$$

similarly
$$\begin{aligned} ca-g^2&=\lambda_2\lambda_3m_1^2+\lambda_3\lambda_1m_2^2+\lambda_1\lambda_2m_3^2,\\ ab-h^2&=\lambda_2\lambda_3n_1^2+\lambda_3\lambda_1n_2^2+\lambda_1\lambda_2n_3^2,\end{aligned}$$

$$\therefore\ bc+ca+ab-f^2-g^2-h^2=\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2.$$

Also
$$\begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix}\begin{vmatrix} \lambda_1l_1, & \lambda_2l_2, & \lambda_3l_3 \\ \lambda_1m_1, & \lambda_2m_2, & \lambda_3m_3 \\ \lambda_1n_1, & \lambda_2n_2, & \lambda_3n_3 \end{vmatrix}$$
$$=\begin{vmatrix} \Sigma\lambda_1l_1^2, & \Sigma\lambda_1l_1m_1, & \Sigma\lambda_1n_1l_1 \\ \Sigma\lambda_1l_1m_1, & \Sigma\lambda_1m_1^2, & \Sigma\lambda_1m_1n_1 \\ \Sigma\lambda_1n_1l_1, & \Sigma\lambda_1m_1n_1, & \Sigma\lambda_1n_1^2 \end{vmatrix},$$

i.e.
$$\lambda_1\lambda_2\lambda_3=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},\quad \text{by § 53.}$$

Therefore the roots of the discriminating cubic are λ_1, λ_2, λ_3. (Cf. § 93.)

The direction-cosines of the new axes can be found as follows: From the above equations we obtain

$$\begin{aligned} al_1+hm_1+gn_1 &= \lambda_1 l_1(l_1^2+m_1^2+n_1^2)+\lambda_2 l_2(l_1 l_2+m_1 m_2+n_1 n_2) \\ &\qquad +\lambda_3 l_3(l_3 l_1+m_3 m_1+n_3 n_1), \\ &= \lambda_1 l_1. \end{aligned}$$

$$\therefore\ (a-\lambda_1)l_1+hm_1+gn_1=0,$$

and similarly,

$$hl_1+(b-\lambda_1)m_1+fn_1=0,$$
$$gl_1+fm_1+(c-\lambda_1)n_1=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Whence l_1, m_1, n_1 and, similarly, the other direction-cosines, satisfy the equations

$$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n}=\lambda, \quad \ldots\ldots\ldots\ldots(5)$$

where λ is the corresponding root of the discriminating cubic. (If we eliminate $l_1 : m_1 : n_1$ from the equations (4), we obtain

$$\begin{vmatrix} a-\lambda_1, & h, & g \\ h, & b-\lambda_1, & f \\ g, & f, & c-\lambda_1 \end{vmatrix}=0,$$

and so another proof that λ_1 is a root of the discriminating cubic.)

Ex. 1. $x^2+3y^2+5z^2-8yz-8xy$ gives, for cubic,

$$\lambda^3-9\lambda^2-9\lambda+81=0, \text{ whence } \lambda=-3,\ 3,\ 9\ ;$$

corresponding to $\lambda_1=-3$, we have $l_1=m_1=2n_1$, or $\dfrac{l_1}{2}=\dfrac{m_1}{2}=\dfrac{n_1}{1}$;

………………… $\lambda_2=\ \ 3$, ………… $-l_2=2m_2=n_2$, or $\dfrac{l_2}{-2}=\dfrac{m_2}{1}=\dfrac{n_2}{2}$;

………………… $\lambda_3=\ \ 9$, ……… $2l_3=-m_3=n_3$, or $\dfrac{l_3}{1}=\dfrac{m_3}{-2}=\dfrac{n_3}{2}$.

Thus if the lines $\dfrac{x}{2}=\dfrac{y}{2}=\dfrac{z}{1}$, $\dfrac{x}{-2}=\dfrac{y}{1}=\dfrac{z}{2}$, $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{2}$ are taken as $O\xi$, $O\eta$, $O\zeta$, respectively, $x^2+3y^2+5z^2-8yz-8xy$ transforms into

$$-3\xi^2+3\eta^2+9\zeta^2.$$

Ex. 2. Verify the following transformations:

(i) $6x^2+11y^2+10z^2-12yz+8zx-4xy$; $3\xi^2+6\eta^2+18\zeta^2$;

$$O\xi,\ \frac{x}{2}=\frac{y}{-1}=\frac{z}{-2};\quad O\eta,\ \frac{x}{2}=\frac{y}{2}=\frac{z}{1};\quad O\zeta,\ \frac{x}{1}=\frac{y}{-2}=\frac{z}{2}.$$

(ii) $4x^2+2y^2+3z^2+4yz-4zx$; $3\eta^2+6\zeta^2$;

$$O\xi,\ \frac{x}{1}=\frac{y}{-2}=\frac{z}{2};\quad O\eta,\ \frac{x}{2}=\frac{y}{2}=\frac{z}{1};\quad O\zeta,\ \frac{x}{-2}=\frac{y}{1}=\frac{z}{2}.$$

(iii) $32x^2+y^2+z^2+6yz-16zx-16xy$; $36\eta^2-2\zeta^2$;

$$\text{O}\xi,\ \frac{x}{1}=\frac{y}{2}=\frac{z}{2};\quad \text{O}\eta,\ \frac{x}{4}=\frac{y}{-1}=\frac{z}{-1};\quad \text{O}\zeta,\ \frac{x}{0}=\frac{y}{1}=\frac{z}{-1}.$$

(iv) $5x^2-16y^2+5z^2+8yz-14zx+8xy$; $12\xi^2-18\eta^2$;

$$\text{O}\xi,\ \frac{x}{-1}=\frac{y}{0}=\frac{z}{1};\quad \text{O}\eta,\ \frac{x}{1}=\frac{y}{-4}=\frac{z}{1};\quad \text{O}\zeta,\ \frac{x}{2}=\frac{y}{1}=\frac{z}{2}.$$

Ex. 3. Reduce to the given forms :

(i) $7x^2+33y^2+7z^2+12yz-10zx-12xy=36$, to $\dfrac{\xi^2}{18}+\dfrac{\eta^2}{4}+\dfrac{\zeta^2}{1}=1$;

(ii) $x^2-31y^2+z^2-20yz-6zx+20xy=36$, to $\dfrac{\xi^2}{4}-\dfrac{\eta^2}{18}-\dfrac{\zeta^2}{1}=1$;

(iii) $2x^2+33y^2+2z^2+12yz-20zx-12xy=72$, to $\dfrac{\xi^2}{8}+\dfrac{\eta^2}{2}-\dfrac{\zeta^2}{9}=1$;

(iv) $13x^2+20y^2+5z^2-12yz+14zx-4xy=336$, to $\dfrac{\xi^2}{24}+\dfrac{\eta^2}{14}=1$;

(v) $5x^2-12y^2-3z^2+20yz-2zx+28xy=336$, to $\dfrac{\xi^2}{24}-\dfrac{\eta^2}{14}=1$.

Ex. 4. Show that

$$\text{(i)}\quad 2x^2+5y^2+10z^2+12yz+6zx+4xy=1$$

and

$$\text{(ii)}\quad 4x^2+y^2-4z^2-12yz-6zx-4xy=1$$

represent an ellipsoid and hyperboloid of revolution, respectively.

(i) transforms into $\xi^2+\eta^2+15\zeta^2=1$ and (ii) into $5\xi^2+5\eta^2-9\zeta^2=1$. In (i) O$\zeta$ is given by $\lambda_3=15$, whence $\dfrac{l_3}{1}=\dfrac{m_3}{2}=\dfrac{n_3}{3}$; $\lambda_1=\lambda_2=1$ give $l_1+2m_1+3n_1=0$; i.e. Oξ and Oη can be any two perpendicular lines in the plane $x+2y+3z=0$, as is evident from the nature of the surface. Derive the same results for the axes in the case of (ii).

Ex. 5. Show that $13x^2+45y^2+40z^2+12yz+36zx-24xy=49$ represents a right circular cylinder whose axis is $x/6=y/2=z/(-3)$ and radius 1.

Consider now the case of

$$ax^2+by^2+cz^2+2fgz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

By the above methods we can determine new axes Oξ, Oη, Oζ, with direction-cosines l_r, m_r, n_r, $r=1, 2, 3$, so that this equation transforms into

$$\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2+2\xi(ul_1+vm_1+wn_1)+2\eta(ul_2+vm_2+wn_2)$$
$$+2\zeta(ul_3+vm_3+wn_3)+d=0,$$

or

$$\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2+2\text{U}\xi+2\text{V}\eta+2\text{W}\zeta+d=0.$$

If λ_1, λ_2, λ_3 are all different from zero, this can be written

$$\lambda_1\left(\xi+\frac{\text{U}}{\lambda_1}\right)^2+\lambda_2\left(\eta+\frac{\text{V}}{\lambda_2}\right)^2+\lambda_3\left(\zeta+\frac{\text{W}}{\lambda_3}\right)^2=\frac{\text{U}^2}{\lambda_1}+\frac{\text{V}^2}{\lambda_2}+\frac{\text{W}^2}{\lambda_3}-d,$$

and if the origin is changed to

$$\left(-\frac{\text{U}}{\lambda_1},\ -\frac{\text{V}}{\lambda_2},\ -\frac{\text{W}}{\lambda_3}\right),$$

this becomes $\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2=\mu$, where

$$\mu=\frac{\mathsf{U}^2}{\lambda_1}+\frac{\mathsf{V}^2}{\lambda_2}+\frac{\mathsf{W}^2}{\lambda_3}-d.$$

If the new origin is (ξ', η', ζ') referred to $\mathsf{O}\xi$, $\mathsf{O}\eta$, $\mathsf{O}\zeta$ and (α, β, γ) referred to OX, OY, OZ, we have the scheme

	α	β	γ
ξ'	l_1	m_1	n_1
η'	l_2	m_2	n_2
ζ'	l_3	m_3	n_3

$$\therefore\ \xi'=-\frac{\mathsf{U}}{\lambda_1}=-\frac{ul_1+vm_1+wn_1}{\lambda_1}=\frac{-l_1(ul_1+vm_1+wn_1)}{al_1+hm_1+gn_1},\text{ by (5),}$$

$$\therefore\ (al_1+hm_1+gn_1)\xi'=-l_1(ul_1+vm_1+wn_1).$$

Similarly $\quad (al_2+hm_2+gn_2)\eta'=-l_2(ul_2+vm_2+wn_2),$

and $\quad (al_3+hm_3+gn_3)\zeta'=-l_3(ul_3+vm_3+wn_3).$

∴ adding, we have $\quad a\alpha+h\beta+g\gamma=-u,$ ....(6)

and, in the same way, $\quad h\alpha+b\beta+f\gamma=-v,$ ....(7)

$\quad g\alpha+f\beta+c\gamma=-w.$ ....(8)

Again,

$$\begin{aligned}\mu&=-\mathsf{U}\xi'-\mathsf{V}\eta'-\mathsf{W}\zeta'-d\\&=-(ul_1+vm_1+wn_1)\xi'-(ul_2+vm_2+wn_2)\eta'-(ul_3+vm_3+wn_3)\zeta'-d\\&=-(u\alpha+v\beta+w\gamma+d).\end{aligned}$$

Thus the values of α, β, γ, μ can be found from the given equation directly. The determinant

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}$$

forms a useful mnemonic for writing down the equations (6), (7), (8) for α, β, γ and the value of μ.

The given equation in this case represents a central conicoid (or cone, if $\mu=0$), and (α, β, γ) is the centre (or vertex).

The above transformation can still be carried out if one of the quantities λ_1, λ_2, λ_3 is zero, provided that the corresponding U or V or W is also zero. Thus if $\lambda_3=\mathsf{W}=0$, the transformed equation is

$$\lambda_1\xi^2+\lambda_2\eta^2+2\mathsf{U}\xi+2\mathsf{V}\eta+d=0,$$

which, by a change of origin, can be reduced to

$$\lambda_1\xi^2+\lambda_2\eta^2=\mu,$$

which represents an elliptic or hyperbolic cylinder, or, if $\mu=0$, a pair of planes.

In this case ζ' is indeterminate. If we multiply the equations (6), (7), (8) by l_3, m_3, n_3, respectively, and add, the sum is identically zero, by (5). Hence (α, β, γ) can be any point on the line of intersection of the planes corresponding to (6) and (7),

$$ax+hy+gz+u=0, \quad hx+by+fz+v=0.$$

This is what is to be expected, since any point on the axis of an elliptic or hyperbolic cylinder can be considered a centre.

Similarly if $\lambda_2=\lambda_3=0$ and $\mathsf{V}=\mathsf{W}=0$, the transformed equation is $\lambda_1\xi^2+2\mathsf{U}\xi+d=0$, which represents a pair of parallel planes; and if $\lambda_2=\lambda_3=0$, $\mathsf{V}\neq 0$, $\mathsf{W}=0$, the transformed equation is

$$\lambda_1\xi^2+2\mathsf{U}\xi+2\mathsf{V}\eta+d=0,$$

which can be reduced to $\lambda_1\xi^2+2\mathsf{V}\eta=0$, and represents a parabolic cylinder. These cases are easily recognisable, as the terms of second degree in the given equation must form a perfect square, and they are easily discussed by a direct method which is indicated in Ex. 6 below.

Lastly, we have the case in which one of the quantities λ_1, λ_2, λ_3 is zero and the corresponding quantity U or V or W is not zero. Suppose $\lambda_3=0$, $\mathsf{W}\neq 0$.

The transformed equation is

$$\lambda_1\xi^2+\lambda_2\eta^2+2\mathsf{U}\xi+2\mathsf{V}\eta+2\mathsf{W}\zeta+d=0,$$

or
$$\lambda_1\left(\xi+\frac{\mathsf{U}}{\lambda_1}\right)^2+\lambda_2\left(\eta+\frac{\mathsf{V}}{\lambda_2}\right)^2+2\mathsf{W}\left(\zeta-\frac{\mathsf{U}^2}{2\mathsf{W}\lambda_1}-\frac{\mathsf{V}^2}{2\mathsf{W}\lambda_2}+\frac{d}{2\mathsf{W}}\right)=0,$$

which, by a change of origin, can be reduced to

$$\lambda_1\xi^2+\lambda_2\eta^2+2\mathsf{W}\zeta=0,$$

and represents an elliptic or hyperbolic paraboloid.

If the final origin is (ξ', η', ζ') referred to $\mathsf{O}\xi$, $\mathsf{O}\eta$, $\mathsf{O}\zeta$ and (α, β, γ) referred to OX, OY, OZ, then

$$\xi'=-\frac{ul_1+vm_1+wn_1}{\lambda_1}=\frac{-l_1(ul_1+vm_1+wn_1)}{al_1+hm_1+gn_1}, \text{ by (5)},$$

and
$$\eta'=-\frac{ul_2+vm_2+wn_2}{\lambda_2}=\frac{-l_2(ul_2+vm_2+wn_2)}{al_2+hm_2+gn_2}.$$

$$\therefore\ (al_1+hm_1+gn_1)\xi'=-l_1(ul_1+vm_1+wn_1),$$
$$(al_2+hm_2+gn_2)\eta'=-l_2(ul_2+vm_2+wn_2),$$

and
$$(al_3+hm_3+gn_3)\zeta'=0, \text{ by (5), since } \lambda_3=0.$$

$\therefore$ adding, we have

$$a\alpha+h\beta+g\gamma=u(l_3^2-1)+vl_3m_3+wl_3n_3,$$
$$=l_3\mathsf{W}-u, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6')$$

and similarly
$$h\alpha+b\beta+f\gamma=m_3\mathsf{W}-v, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7')$$

and
$$g\alpha+f\beta+c\gamma=n_3\mathsf{W}-w. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8')$$

These equations correspond to (6), (7), (8), but are not independent, since, if we multiply them by l_3, m_3, n_3 respectively and add, the result is identically zero. But we have also

$$2\mathsf{W}\zeta' = \frac{\mathsf{U}^2}{\lambda_1} + \frac{\mathsf{V}^2}{\lambda_2} - d,$$

or $2(ul_3 + vm_3 + wn_3)\zeta' = -(ul_1 + vm_1 + wn_1)\xi' - (ul_2 + vm_2 + wn_2)\eta' - d,$

$$\therefore\ \mathsf{W}\zeta' = -u\alpha - v\beta - w\gamma - d,$$

or $$\mathsf{W}(l_3\alpha + m_3\beta + n_3\gamma) = -u\alpha - v\beta - w\gamma - d. \qquad \text{.................(9)}$$

Any two of the equations (6′), (7′), (8′), with equation (9), determine the values of α, β, γ, which are the coordinates of the vertex of the paraboloid. If we multiply the equations (6′), (7′), (8′) by α, β, γ, respectively, and add to (9), we obtain

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d = 0,$$

thus verifying that the point (α, β, γ) lies on the surface. Note that in this case the three equations (6), (7), (8), i.e. $a\alpha + h\beta + g\gamma + u = 0$ $h\alpha + b\beta + f\gamma + v = 0$, $g\alpha + f\beta + c\gamma + w = 0$, give, when multiplied by l_3, m_3, n_3, respectively, and added, $ul_3 + vm_3 + wn_3 = \mathsf{W} = 0$, since, by (5), $al_3 + hm_3 + gn_3 = \lambda_3 l_3 = 0$, etc. But, by hypothesis, $\mathsf{W} \neq 0$, so that the equations (6), (7), (8) give a point at infinity. (Cf. § 45.)

We may sum up the methods of reduction in the various cases as follows :

(1) If the terms of second degree form a perfect square proceed as in Ex. 6 below.

In any other case, first solve the discriminating cubic.

(2) If the roots λ_1, λ_2, λ_3 are all different from zero, find (α, β, γ) from the equations (6), (7), (8), i.e. $a\alpha + h\beta + g\gamma + u = 0$, $h\alpha + b\beta + f\gamma + v = 0$, $g\alpha + f\beta + c\gamma + w = 0$. Then $\mu = -(u\alpha + v\beta + w\gamma + d)$ and the reduced equation is $\lambda_1\xi^2 + \lambda_2\eta^2 + \lambda_3\zeta^2 = \mu$.

(3) If one root, λ_3, say, is zero ; attempt to solve the equations (6), (7), (8). If they reduce to two independent equations, assign an arbitrary value to one of the quantities α, β, γ, and solve for the other two ; e.g. put $\gamma = 0$. Then $\mu = -(u\alpha + v\beta + w\gamma + d)$ and the reduced equation is $\lambda_1\xi^2 + \lambda_2\eta^2 = \mu$.

If the equations lead to a paradoxical result (as in § 45, p. 47), find l_3, m_3, n_3 from any two of the equations

$$al_3 + hm_3 + gn_3 = 0, \quad hl_3 + bm_3 + fn_3 = 0, \quad gl_3 + fm_3 + cn_3 = 0.$$

Then $\mathsf{W} = ul_3 + vm_3 + wn_3$, and the reduced equation is

$$\lambda_1\xi^2 + \lambda_2\eta^2 + 2\mathsf{W}\zeta = 0.$$

Ex. 6. Reduce the equations :

$$\text{(i)}\ (3x - 4y + z)^2 + 9x - 12y + 3z - 10 = 0,$$
$$\text{(ii)}\ (3x - 4y + z)^2 = 26(x + y + z),$$
$$\text{(iii)}\ (3x - 4y + z)^2 + 16x - 27y + 18 = 0.$$

Equation (i) can be written

$$(3x - 4y + z)^2 + 3(3x - 4y + z) - 10 = 0,$$

or
$$(3x-4y+z+5)(3x-4y+z-2)=0,$$
and therefore represents a pair of parallel planes.

In (ii) we note that $3x-4y+z=0$, $x+y+z=0$ represent perpendicular planes. Taking the first as $\zeta O\xi$ and the second as $\eta O\zeta$, we have $\eta=\dfrac{3x-4y+z}{\sqrt{26}}$, $\xi=\dfrac{x+y+z}{\sqrt{3}}$, so that (ii) reduces to $\eta^2=\sqrt{3}\xi$, and represents a parabolic cylinder.

Write (iii) in the form
$$(3x-4y+z+\lambda)^2=x(6\lambda-16)-y(8\lambda-27)+2\lambda z+\lambda^2-18,$$
and choose λ so that the planes
$$3x-4y+z+\lambda=0, \quad x(6\lambda-16)-y(8\lambda-27)+2\lambda z+\lambda^2-18=0$$
are at right angles. This gives $\lambda=3$, and the equation becomes
$$(3x-4y+z+3)^2=2x+3y+6z-9,$$
which, as in (ii), reduces to $26\eta^2=7\xi$.

Ex. 7. Verify the following reductions:

(i) $(2x-2y-3z)^2+19x-34y-44z+50=0$; $17\eta^2=11\xi$.

(ii) $5x^2+7y^2+6z^2-4yz-4zx-6x-10y-4z+7=0$;
$\xi^2+2\eta^2+3\zeta^2=1$, centre $(1, 1, 1)$.

(iii) $7x^2+y^2+z^2+16yz+8zx-8xy+2x+4y-40z-14=0$;
$\xi^2+\eta^2-\zeta^2=1$, centre $(1, 2, 0)$.

(iv) $2x^2-y^2-10z^2+20yz-8zx-28xy+16x+26y+16z-34=0$;
$2\xi^2-\eta^2-2\zeta^2=1$, centre $(\frac{13}{9}, \frac{5}{9}, \frac{7}{9})$.

(v) $11x^2+5y^2+2z^2+20yz+4zx-16xy-10x-14y-28z+26=0$;
$2\xi^2+\eta^2-\zeta^2=0$, vertex $(1, 1, 1)$.

(vi) $4x^2+3y^2+2z^2+4yz-4xy-4x-6y-8z+6=0$; $\xi^2+2\eta^2=1$,
axis $\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-1}{-2}$.

(vii) $x^2-y^2+4yz+4zx-6x-2y-8z+5=0$; $\xi^2-\eta^2=1$,
axis $\dfrac{x-1}{-2}=\dfrac{y-1}{2}=\dfrac{z-1}{1}$.

(viii) $32x^2+y^2+z^2+6yz-16zx-16xy-6x-12y-12z+18=0$;
$18\xi^2-\eta^2=9\zeta$, vertex $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

(ix) $5x^2-16y^2+5z^2+8yz-14zx+8xy+4x+20y+4z-24=0$;
$2\xi^2-3\eta^2+2\zeta=0$, vertex $(1, 1, 1)$.

Ex. 8. Reduce the equation:
$$x^2+2y^2+3z^2-6yz+4zx-2xy-10x+12y-10z+7=0.$$

The discriminating cubic is $\lambda^3-6\lambda^2-3\lambda+2=0$, of which the roots cannot be found by inspection. There are two changes in the sequence of signs in the equation, and hence, by Descartes' Rule, two of the roots are positive and one is negative. (The values to three places are $-0{\cdot}806$, $0{\cdot}387$, $6{\cdot}419$.) We find in the usual way that $\alpha=1$, $\beta=2$, $\gamma=3$, $\mu=1$. The reduced equation is $-0{\cdot}806\xi^2+0{\cdot}387\eta^2+6{\cdot}419\zeta^2=1$, and represents a hyperboloid of one sheet.

MISCELLANEOUS EXAMPLES I.

1. If p is the distance of the origin from a line whose direction-cosines are l, m, n, and p_1, p_2, p_3 are the distances of the origin from the projections of the line on the planes YOZ, ZOX, XOY, respectively, show that

$$p_1^2+p_2^2+p_3^2-p^2=l^2p_1^2+m^2p_2^2+n^2p_3^2.$$

2. The equations of the lines bisecting the angles between the lines $\frac{x}{2}=\frac{y}{-1}=\frac{z}{0}$; $\frac{x}{-10}=\frac{y}{1}=\frac{z}{12}$; are $\frac{x}{2}=\frac{y}{-3}=\frac{z}{6}$; $\frac{x}{-6}=\frac{y}{2}=\frac{z}{3}$.

3. Show that the point $(2, 3, -1)$ lies on both the planes

$$2x-y-z=2, \quad x+2y-3z=11,$$

and that the line of intersection of the planes is given by

$$x-2=y-3=z+1.$$

4. The equations of the perpendicular from the point $(5, 9, 3)$ to the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ are $\frac{x-5}{1}=\frac{y-9}{2}=\frac{z-3}{-2}$, and the foot of the perpendicular is $(3, 5, 7)$.

5. If OZ is vertical, the line of greatest slope through the point $(1, 1, 1)$ on the plane $2x+3y+4z=9$ is given by

$$\frac{x-1}{8}=\frac{y-1}{12}=\frac{z-1}{-13}.$$

6. The orthogonal projection of the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ on the plane $3x+4y+5z=0$ is $\frac{x}{7}=\frac{y}{1}=\frac{z}{-5}$.

7. Show that the lines

$$\text{(i)}\quad x+y+z-3=0=2x+3y+4z-5,$$
$$4x-y+5z-7=0=2x-5y-z-3;$$

$$\text{(ii)}\quad \frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3},\quad \frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4};$$

are coplanar, and lie in the planes

(i) $x+2y+3z=2$, (ii) $x-2y+z+7=0$, respectively.

8. If $a+b+c=0$, show that the line $x+a=y+b=z+c$ intersects the four lines $x=0$, $y+z=3a$; $y=0$, $z+x=3b$; $z=0$, $x+y=3c$;

$x+y+z=3k$, $\dfrac{ax}{k-a}+\dfrac{by}{k-b}+\dfrac{cz}{k-c}=0$; at right angles, and that the shortest distances between the first of these four lines and the other three are

$$\sqrt{3}(a-b),\quad \sqrt{3}(a-c),\quad \sqrt{3}(a-k).$$

9. Find the equations of the common perpendicular to the lines

$$x=0,\ y+z=b-c;\quad y=0,\ z+x=c-a;$$

in the form

$$3x+b-c=3y+c-a=3z+a-b.$$

Show that it is also at right angles to the line $z=0$, $x+y=a-b$, and that the lengths of the shortest distances between the three lines are in the ratios

$$b+c-2a : c+a-2b : a+b-2c.$$

10. If $pl+qm+rn=0$, show that each of the sets of equations

$$p-ny+mz=0;\quad q-lz+nx=0;\quad r-mx+ly=0;\quad \text{.........(i)}$$

$$l-ry+qz=0;\quad m-pz+rx=0;\quad n-qx+py=0;\quad \text{.........(ii)}$$

is satisfied at all points of one of two perpendicular lines; that the length and equations of their shortest distance are

$$\frac{l^2+m^2+n^2+p^2+q^2+r^2}{(l^2+m^2+n^2)^{\frac{1}{2}}(p^2+q^2+r^2)^{\frac{1}{2}}};\quad lx+my+nz=0=px+qy+rz;$$

and that the shortest distance is divided by the origin in the ratio

$$(l^2+m^2+n^2):(p^2+q^2+r^2).$$

11. Show that the three planes $bz-cy=b-c$, $cx-az=c-a$, $ay-bx=a-b$, pass through one line, L, and that the three planes

$$(c-a)z-(a-b)y=b+c,\quad (a-b)x-(b-c)z=c+a,$$

$$(b-c)y-(c-a)x=a+b,$$

pass through a line L′; that L and L′ are at right angles and that the shortest distance between them is of length

$$2(a^2+b^2+c^2)^{\frac{1}{2}}/[(b-c)^2+(c-a)^2+(a-b)^2]^{\frac{1}{2}}$$

and has the equations

$$ax+by+cz=a+b+c,\quad (b-c)x+(c-a)y+(a-b)z=0;$$

and that the shortest distance meets L and L′ in the points given by $x=y=z=1$,

$$\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}=\frac{1}{a^2+b^2+c^2-bc-ca-ab}.$$

12. Show that the equations of any line which meets the three lines

$$x=a,\ y=0;\quad y=a,\ z=0;\quad z=a,\ x=0;$$

can be written

$$x-a=ty,\quad z(t+1)+t(y-a)=0,\quad \text{....................(i)}$$

and that the locus of such lines is the surface

$$yz+zx+xy-ax-ay-az+a^2=0.\quad \text{....................(ii)}$$

Show that the equations of any line which meets the three lines

$$z=a,\ y=0;\quad x=a,\ z=0;\quad y=a,\ x=0;$$

can be written

$$x-a=sz,\quad y(s+1)+s(z-a)=0,\quad \text{..................(iii)}$$

and that this line also lies on the surface (ii).

Show that the lines (i) and (iii) meet at a point P given

$$\frac{x}{(t+1)(s+1)}=\frac{y}{s}=\frac{z}{t}=\frac{a}{s+t+1},$$

and that, if they meet at right angles, either $st=1$, or $st+s+t=0$. Deduce that, in these cases, P lies on the sphere

$$x(x-a)+y(y-a)+z(z-a)=0$$

and on one of the planes $x+y+z=a$, $x+y+z=2a$.

13. Show that the length of the S.D. between the lines

$$\frac{x+3}{2}=\frac{y-6}{3}=\frac{z-3}{-2},\quad \frac{x}{2}=\frac{y-6}{2}=\frac{z}{-1};$$

is 3 and that the line of the S.D. cuts OZ. Show also that all lines which cut the given lines and OZ lie on the surface

$$7(x-y+6)(x+z)=(3x-2y+21)(x+2z).$$

14. Show that the condition that the line drawn through the origin to meet both of the lines

$$\frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1},\quad \frac{x-a_2}{l_2}=\frac{y-b_2}{m_2}=\frac{z-c_2}{n_2},$$

should also meet the line $\dfrac{x-a_3}{l_3}=\dfrac{y-b_3}{m_3}=\dfrac{z-c_3}{n_3}$ is

$$\begin{vmatrix} a_3, & b_3, & c_3 \\ a_1, & b_1, & c_1 \\ l_1, & m_1, & n_1 \end{vmatrix} \cdot \begin{vmatrix} l_3, & m_3, & n_3 \\ a_2, & b_2, & c_2 \\ l_2, & m_2, & n_2 \end{vmatrix} = \begin{vmatrix} a_3, & b_3, & c_3 \\ a_2, & b_2, & c_2 \\ l_2, & m_2, & n_2 \end{vmatrix} \cdot \begin{vmatrix} l_3, & m_3, & n_3 \\ a_1, & b_1, & c_1 \\ l_1, & m_1, & n_1 \end{vmatrix}.$$

15. Show that the lines

$$u_1\equiv a_1x+b_1y+c_1z+d_1=0,\quad u_2\equiv a_2x+b_2y+c_2z+d_2=0;$$
$$u_3\equiv a_3x+b_3y+c_3z+d_3=0,\quad u_4\equiv a_4x+b_4y+c_4z+d_4=0;$$

are coplanar if

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_2, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}=0,$$

and that the equation of the plane containing them can be written

$$\begin{vmatrix} a_1, & b_1, & c_1, & u_1 \\ a_2, & b_2, & c_2, & u_2 \\ a_3, & b_3, & c_3, & 0 \\ a_4, & b_4, & c_4, & 0 \end{vmatrix}=0,\quad\text{or}\quad \begin{vmatrix} a_1, & b_1, & c_1, & 0 \\ a_2, & b_2, & c_2, & 0 \\ a_3, & b_3, & c_3, & u_3 \\ a_4, & b_4, & c_4, & u_4 \end{vmatrix}=0.$$

16. If the equal and parallel edges of a triangular prism pass through the points (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) and the lengths of their projections on OX, OY, OZ are α, β, γ, respectively, show that the volume of the prism is

$$\frac{1}{2}\begin{vmatrix} \alpha, & \beta, & \gamma, & 0 \\ a_1, & b_1, & c_1, & 1 \\ a_2, & b_2, & c_2, & 1 \\ a_3, & b_3, & c_3, & 1 \end{vmatrix}.$$

17. The plane $x+y+z=1$ cuts the axes OX, OY, OZ, which are rectangular, in A, B, C. P is the point (α, β, γ) and the feet of the perpendiculars from P to OA, BC ; OB, CA ; OC, AB, are D, D′ ; E, E′ ; F, F′, respectively. Show that the equations of DD′ are

$$\frac{x-\alpha}{2\alpha}=\frac{y}{-\beta+\gamma-1}=\frac{z}{-\gamma+\beta-1}.$$

Show also that if DD′, EE′, FF′ are concurrent, P lies on the line $x=y=z$, or on one of the hyperbolas

$$y=z,\ 2zx-z-x+1=0;\quad z=x,\ 2xy-x-y+1=0;$$
$$x=y,\quad 2yz-y-z+1=0.$$

18. The lines AB, CD, EF are given by the equations $y=b,\ z=-c$; $z=c,\ x=-a$; $x=a,\ y=-b$. If P is a point such that the planes PAB, PCD intercept a constant length $2k$ on EF, show that P lies on one of the hyperboloids

$$ayz+bzx+cxy+abc=\pm k(x+a)(y-b).$$

19. A variable sphere passes through the points $(0, 0, \pm c)$ and cuts the lines $x\sin\alpha=\pm y\cos\alpha,\quad z=\pm c$ in the points P and P′. If PP′ has a constant length $2k$, show that the centre of the sphere lies on the circle $z=0,\ x^2+y^2=(k^2-c^2)\operatorname{cosec}^2 2\alpha$.

20. A variable line which meets $y=mx,\ z=c$; and $y=-mx,\ z=-c$; and which intersects the ellipse $x=0,\ y^2/a^2+z^2/b^2=1$, generates the surface

$$\frac{(y^2-m^2x^2)^2}{a^2}+\frac{(yz-cmx)^2}{b^2}=\frac{(cy-mzx)^2}{c^2}.$$

21. Find the equations of the line through (α, β, γ) which meets both of the lines $y=mx,\ z=c$; $y=-mx,\ z=-c$; in the forms

$$\text{(i)}\ \frac{y-mx}{\beta-m\alpha}=\frac{z-c}{\gamma-c},\quad \frac{y+mx}{\beta+m\alpha}=\frac{z+c}{\gamma+c};$$

$$\text{(ii)}\ \frac{m(x-\alpha)}{m\gamma\alpha-c\beta}=\frac{y-\beta}{\beta\gamma-mc\alpha}=\frac{z-\gamma}{\gamma^2-c^2}.$$

22. The locus of a variable line which is always parallel to OX and intersects the given lines

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},\quad \frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'},$$

is given by the equation

$$\begin{vmatrix} x-a, & y-b \\ l, & m \end{vmatrix}\begin{vmatrix} x-a', & z-c' \\ l', & n' \end{vmatrix}=\begin{vmatrix} x-a', & y-b' \\ l', & m' \end{vmatrix}\begin{vmatrix} x-a, & z-c \\ l, & n \end{vmatrix}.$$

23. Show that the tetrahedron whose vertices are (1, 2, 3), (6, 4, 2), (3, −7/2, 8), (−5/8, 15/4, 35/8) is self-polar with respect to the sphere $x^2+y^2+z^2=20$, and verify that each edge is at right angles to the opposite edge and that the planes through the edges at right angles to the opposite edges all pass through the centre of the sphere.

24. Show that A, (1, 2, 3), B, (0, 11, 7), C, (−3, 2, 11), D, (−8, 5, 3) are the vertices of a tetrahedron in which each edge is at right angles to the opposite edge. Show that the perpendiculars from the vertices to the opposite faces meet at the point P, $\left(-\frac{31}{14}, \frac{61}{14}, \frac{165}{28}\right)$ and that P divides the shortest distances between the pairs of edges AB, CD; AC, BD; AD, BC; in the ratios 31 : 33, 319 : 465, 1431 : 1705, respectively. Verify that the mid-points of the edges lie on the sphere $x^2+y^2+z^2+5x-10y-12z+55=0$, whose centre is the centroid of the tetrahedron.

25. If O is the origin and A, B, C are the points $(4a, 4b, 4c)$, $(4b, 4c, 4a)$, $(4c, 4a, 4b)$, show that the sphere

$$x^2+y^2+z^2-2(x+y+z)(a+b+c)+8(bc+ca+ab)=0$$

passes through the nine-point circles of the faces of the tetrahedron OABC.

26. For the tetrahedron whose vertices are

A, $(1, -2, 11)$, B, $(-4, 2, 4)$, C, $(3, -6, 6)$, D, $(2, 2, 1)$,

obtain the following results:

The faces are $16x+13y-4z+54=0$, $8x+11y+16z-54=0$,

$$20x+5y+4z-54=0, \quad 4x+19y+8z-54=0;$$

the area of each face being 63/2:

The lines joining the mid-points of pairs of opposite edges are mutually perpendicular and the lengths of the S.D. between the opposite edges are 3, 6, 9:

The volume of the tetrahedron is 54:

The equation of the circumscribing sphere is

$$x^2+y^2+z^2-x+2y-11z=0,$$

and the three faces which meet at any vertex make the same angles with the tangent plane to the sphere at that vertex:

The centre of the inscribed sphere is $(\frac{1}{2}, -1, \frac{11}{2})$ and its radius is 9/7; the centres of the escribed spheres are $(0, 0, 0)$, $(5, -4, 7)$, $(-2, 4, 5)$, $(-1, -4, 10)$, and their radii are all equal to 18/7.

27. Show that the planes $x-y-z-9=0$, $5x+y+z-9=0$, $x+5y-z-15=0$, $x-y+5z-21=0$, are the faces of a regular tetrahedron which is inscribed in the sphere

$$x^2+y^2+z^2-9x+y-z-40=0$$

and circumscribed to the sphere

$$x^2+y^2+z^2-9x+y-z+14=0.$$

Show that the escribed spheres are

$$x^2+y^2+z^2+6x+4y+2z-13=0, \quad x^2+y^2+z^2-12x-14y+2z+59=0;$$
$$x^2+y^2+z^2-12x+4y-16z+77=0, \quad x^2+y^2+z^2-18x+10y+8z+95=0.$$

Show also that the spheres which touch all the edges are

$$x^2+y^2+z^2-9x+y-z+\tfrac{1}{2}=0,$$
$$x^2+y^2+z^2+21x+7y+5z-\tfrac{107}{2}=0,$$
$$x^2+y^2+z^2-15x+7y-31z+\tfrac{253}{2}=0,$$
$$x^2+y^2+z^2-15x-29y+5z+\tfrac{181}{2}=0,$$
$$x^2+y^2+z^2-27x+19y+17z+\tfrac{325}{2}=0.$$

28. The six spheres

$$x^2+y^2+z^2-2x+22z-4=0,$$
$$x^2+y^2+z^2+8x+2z-4=0,$$
$$x^2+y^2+z^2-12x+42z+36=0,$$
$$x^2+y^2+z^2-12x-18z+36=0,$$
$$4x^2+4y^2+4z^2+42x-12z+9=0,$$
$$4x^2+4y^2+4z^2-18x-12z+9=0$$

all touch the four sides of the skew quadrilateral given by

$$x=0,\quad \pm y+2z=3\;;\quad z=0,\ x\pm 2y=6.$$

29. If the spheres
$$\mathsf{S}_1\equiv x^2+y^2+z^2+2a_1x+2b_1y+2c_1z+d_1=0,$$
$$\mathsf{S}_2\equiv x^2+y^2+z^2+2a_2x+2b_2y+2c_2z+d_2=0,$$
meet in a great circle on the sphere S_1,
$$2a_1^2+2b_1^2+2c_1^2-d_1=2a_1a_2+2b_1b_2+2c_1c_2-d_2.$$

30. The centres of spheres which cut both the spheres
$$x^2+y^2+z^2+2u_1x+d=0,\quad x^2+y^2+z^2+2u_2x+d=0$$
in great circles lie on the plane $x+u_1+u_2=0$.

31. Given the coaxal spheres
$$\mathsf{S}_1\equiv x^2+y^2+z^2+2u_1x+d=0,$$
$$\mathsf{S}_2\equiv x^2+y^2+z^2+2u_2x+d=0,$$
$$\mathsf{S}_3\equiv x^2+y^2+z^2+2(2u_2-u_1)x+d=0,$$
show that S_3 cuts in a great circle any sphere orthogonal to S_1 which has its centre on S_2, and that S_3 cuts orthogonally any sphere which has its centre on S_2 and is cut by S_1 in a great circle.

32. Show that the circles
$$x^2+y^2+z^2-4x+4y-2z-7=0,\ 2x-2y+z-9=0\;;$$
$$x^2+y^2+z^2-2x-3y+6z+3=0,\ 2x+3y-6z-28=0,$$
lie on the same sphere and find its equation.

33. Lines are drawn through O to meet the circle in which the plane (i) $x+y+z=1$ cuts the sphere $x^2+y^2+z^2-4x-6y-8z+4=0$. Show that they lie on the cone $x^2-y^2-3z^2-6yz-4zx-2xy=0$ and meet the sphere again at points on the plane, (ii) $y+2z=2$. Show that
$$t(x^2+y^2+z^2-4x-6y-8z+4)+2(y+2z-2)(x+y+z-1)=0$$
represents a cone if $t=-1$ or if $t=19/25$, and deduce that the circles in which the planes (i) and (ii) cut the given sphere lie on a second cone whose vertex is $(-36/7,\ -4/7,\ 4)$.

34. Show that two cones can be drawn through the circles in which the planes $x+y+z=1$, $y+2z=3$ cut the sphere $x^2+y^2+z^2=4$, and that their vertices are $(1,\ 2,\ 3)$ and $(-2,\ 0,\ 2)$.

35. If lines are drawn through points on the circle in which the plane $lx+my+nz=k$ cuts the sphere $x^2+y^2+z^2=a^2$ parallel to the line $x/p=y/q=z/r$, show that they meet the sphere again in points on the plane
$$(lx+my+nz-k)(p^2+q^2+r^2)=2(px+qy+rz)(pl+qm+rn).$$

36. Of a set of rectangular axes through O, one lies in the plane $ax+by+cz=0$, and one in the plane $px+qy+rz=0$. Show that the third lies on the cone

$$\frac{1}{ap}\left(\frac{y}{b}-\frac{z}{c}\right)\left(\frac{y}{q}-\frac{z}{r}\right)+\frac{1}{bq}\left(\frac{z}{c}-\frac{x}{a}\right)\left(\frac{z}{r}-\frac{x}{p}\right)+\frac{1}{cr}\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{p}-\frac{y}{q}\right)=0.$$

37. If OX, OY, OZ, Oξ, Oη, Oζ are two sets of rectangular axes through the same origin O, show that the equation of the cone which passes through them can be written in the forms

$$\frac{l_1l_2l_3}{x}+\frac{m_1m_2m_3}{y}+\frac{n_1n_2n_3}{z}=0\;;\quad \frac{l_1m_1n_1}{\xi}+\frac{l_2m_2n_2}{\eta}+\frac{l_3m_3n_3}{\zeta}=0\;;$$

where l_r, m_r, n_r, $r=1, 2, 3$, are the direction-cosines of Oξ, Oη, Oζ, referred to OX, OY, OZ. Deduce that

$$l_1{}^2l_2{}^2l_3{}^2+m_1{}^2m_2{}^2m_3{}^2+n_1{}^2n_2{}^2n_3{}^2=l_1{}^2m_1{}^2n_1{}^2+l_2{}^2m_2{}^2n_2{}^2+l_3{}^2m_3{}^2n_3{}^2.$$

38. A, B, C, D, E are the points (1, 2, 3), (3, 2, 7), (4, 1, 9), (8, 11, 21), (4, 19, 21), respectively. Show that the equation

$$(11x+y-5z)(9x+3y-5z)+t(4x+y-2z)(13x+2y-6z)=0$$

represents a cone passing through the lines OA, OB, OC, OD, where O is the origin. Find the value of t if the cone passes through E and deduce that the cone through the five lines is given by

$$5x^2+y^2-z^2=0.$$

39. Show that the cone generated by rotating the line $x/l=y/m=z/n$ about the line $x/a=y/b=z/c$ as axis is given by

$$(l^2+m^2+n^2)(ax+by+cz)^2=(al+bm+cn)^2(x^2+y^2+z^2).$$

40. If α is the semi-vertical angle of the right circular cone which passes through the lines OX, OY, $x=y=z$, show that

$$\cos\alpha=(9-4\sqrt{3})^{-\frac{1}{2}}.$$

41. The plane $x+y+z=0$ cuts the cone $5x^2-32y^2+27z^2=0$ in perpendicular lines. Show that the cone through these lines and OX, OY, OZ is given by $5yz-32zx+27xy=0$. Show also that the planes through the origin at right angles to these lines are given by

$$-32x+5y+27z=\pm\sqrt{889}(x-y),\ \text{or}\ \ -27x+32y-5z=\pm\sqrt{889}(x-z).$$

42. If l, m, n, and l', m', n' are the direction-cosines of the lines in which the plane $px+qy+rz=0$ cuts the cone

$$\frac{(b-c)p}{x}+\frac{(c-a)q}{y}+\frac{(a-b)r}{z}=0,$$

show that

$$\frac{ll'}{b-c}=\frac{mm'}{c-a}=\frac{nn'}{a-b}\,;$$

then show, by taking the given plane as a new coordinate plane, and changing the axes, that these lines bisect the angles between the lines in which the given plane cuts the cone $ax^2+by^2+cz^2=0$.

43. An ellipse of semi-axes a and b has its centre at the origin and the direction-cosines of its axes are respectively l_1, m_1, n_1; and l_2, m_2, n_2. Show that the ellipse itself is the orthogonal projection on its plane of

$$\frac{(l_1x+m_1y)^2}{a^2}+\frac{(l_2x+m_2y)^2}{b^2}=1, \quad z=0;$$

and that its orthogonal projection on the plane XOY is given by

$$\frac{(m_2x-l_2y)^2}{a^2}+\frac{(m_1x-l_1y)^2}{b^2}=(l_1m_2-l_2m_1)^2.$$

44. Show that the section of the cone $x^2+y^2-2z^2=0$ by the plane $x+y-2z=1$ is a parabola whose equations can be written $Z=0$, $6X^2+4\sqrt{3}Y=1$, and that the section of the hyperboloid $x^2+y^2-z^2=1$ by the plane $x+y-2z=0$ is an ellipse whose equations can be written $Z=0$, $X^2+Y^2/3=1$.

45. What does the equation $x^2+y^2+7z^2+8yz-8zx+16xy=81$ become when the lines joining the origin to the points (1, 2, 2), (2, 1, -2), (2, -2, 1) are taken as coordinate axes?

46. The cone whose vertex is (1, 2, 3) and which passes through the circle

$$x^2+y^2+z^2=4, \quad x+y+z=1,$$

is given by

$$2(x+y+z-1)(y+2z-3)=5(x^2+y^2+z^2-4).$$

47. Show that the two cones $ayz+bzx+cxy=0$, $fyz+gzx+hxy=0$ intersect in the coordinate axes and in the line

$$x(bh-cg)=y(cf-ah)=z(ag-bf).$$

48. The points common to the two cones $y^2-z^2+3yz-2zx+xy=0$, $x^2-z^2-2yz+3zx+xy=0$, lie in the pairs of planes

$$(x-y)(x+y+5z)=0, \quad (x+y-z)(x+y+2z)=0.$$

49. The cone through OX, OY, OZ and the lines in which the plane $a(b-c)x+b(c-a)y+c(a-b)z=0$ cuts the cone

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0$$

is given by $(b-c)yz+(c-a)zx+(a-b)xy=0$, and the other two common generators of the two cones lie in the plane

$$a(b+c)x+b(c+a)y+c(a+b)z=0.$$

50. The common generators of the two cones

$$(b+c)(b-c)^3x^2+(c+a)(c-a)^3y^2+(a+b)(a-b)^3z^2=0,$$

$ayz+bzx+cxy=0$, lie in the planes

$$(b-c)^2x+(c-a)^2y+(a-b)^2z=0, \quad (b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z=0.$$

51. Show that

$$-x+2y+2z=0, \quad 2x-y+2z=0, \quad 2x+2y-z=0;$$
$$6x-2y+3z=0, \quad 3x+6y-2z=0, \quad -2x+3y+6z=0,$$

are two sets of mutually perpendicular tangent planes to the cone

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

and that, in each case, the generators of contact are mutually inclined at angles of 60°.

52. Show that the cone

$$\frac{x^2}{b-c}+\frac{y^2}{c-a}+\frac{z^2}{a-b}=0$$

has sets of three mutually perpendicular tangent planes. If one plane of a set is $x+y+z=0$, show that the other two are given by

$$(b-c)(x^2+2yz)+(c-a)(y^2+2zx)+(a-b)(z^2+2xy)=0,$$

and that the line of intersection of these planes is $x=y=z$. Show also that the lines along which these planes touch the cone lie in the plane

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0.$$

53. Show that, for all values of t, the plane

$$at^2(x-y-z)-2t(bx+ay)-b(x-y+z)=0$$

cuts the cone $ax^2+by^2=(a+b)z^2$ in perpendicular lines, and deduce that all such planes touch the cone

$$\frac{x^2}{a}+\frac{y^2}{b}=\frac{z^2}{a+b}.$$

54. Show that any cone whose vertex is on the paraboloid $x^2+y^2=2cz$ and which passes through a meridian section has sets of three mutually perpendicular generators.

55. A cone has vertex $(0, 0, c)$ and base $z=0$, $y^2=4ax$. Show that its equation is $cy^2+4ax(z-c)=0$ and that the equation of the reciprocal cone is $ay^2+cx(z-c)=0$.

56. The cone whose vertex is $(0, 0, 2a)$ and base $z=0$, $y^2=4ax$ is its own reciprocal. The plane $z=x$ cuts it in a circle. By changing the origin to the vertex, and turning the x- and z-axes through 45°, show that the cone is right circular and has a semi-vertical angle of 45°. Draw a diagram showing the cone and the section by the plane $z=x$.

57. The cone whose vertex is $(0, 0, c)$ and base $z=0$, $x^2+y^2-2ax=0$ has a reciprocal cone given by $a^2y^2=c(z-c)(cz-c^2-2ax)$. Hence show that if the section of a cone by a plane at right angles to a generator is a circle, the section of the reciprocal cone by the same plane is a parabola.

58. Show that the cone whose vertex is $(3, 4, 5)$ and which passes through the curve of intersection of the plane $x+y+z=0$ and the surface $yz+zx+xy+1=0$ is given by

$$5x^2+4y^2+3z^2-5yz-4zx-3xy+2x+2y+2z-12=0$$

and that it meets the surface again at points in the plane

$$5x+4y+3z+2=0.$$

59. Lines drawn through points on the circle

$$x+y+z=1,\quad x^2+y^2+z^2=4,$$

parallel to the line $x/2=y/-1=z/2$ generate a cylinder whose equation is

$$2x^2+8y^2+2z^2+4yz-2zx+4xy-2x-8y-2z-9=0.$$

The cylinder meets the sphere in a second circle in the plane

$$x-5y+z+3=0$$

and the two circles lie on the cone

$$25x^2+y^2+25z^2-16yz+8zx-16xy+8x+32y+8z-96=0$$

whose vertex is at the point (4/3, 16/3, 4/3).

60. Show that the equations of the two planes which pass through the line $2x=-y=2z$ and cut the cone $15x^2-14y^2+5z^2=0$ in perpendicular generators are $3x+2y+z=0$, $2x-y-4z=0$.

61. If two planes, each of which cuts the cone $ax^2+by^2+cz^2=0$ in perpendicular lines, are at right angles, their line of intersection lies on the cone $(2a+b+c)x^2+(2b+c+a)y^2+(2c+a+b)z^2=0$.

62. Find the tangent planes to the hyperboloid $3x^2-7y^2+42z^2=126$ which pass through the lines

$$\text{(i)}\ \frac{x-8}{-7}=\frac{y+4}{5}=\frac{z-1}{1},\quad \text{and}\quad \text{(ii)}\ 3x+7y+4z=0=x+4y+3z+3,$$

giving the points of contact in each case.

63. Show that the pair of tangent planes to $ax^2+by^2+cz^2=1$ whose line of intersection is $\dfrac{x-p}{l}=\dfrac{y-q}{m}=\dfrac{z-r}{n}$ is given by

$$\frac{\begin{vmatrix} y-q, & z-r \\ m, & n \end{vmatrix}^2}{a}+\frac{\begin{vmatrix} z-r, & x-p \\ n, & l \end{vmatrix}^2}{b}+\frac{\begin{vmatrix} x-p, & y-q \\ l, & m \end{vmatrix}^2}{c}=\begin{vmatrix} x, & y, & z \\ l, & m, & n \\ p, & q, & r \end{vmatrix}^2,$$

and deduce that, if (p, q, r) is a fixed point and the planes are at right angles, the line of intersection lies on the cone

$$a(b+c)(x-p)^2+b(c+a)(y-q)^2+c(a+b)(z-r)^2$$
$$=abc[(ry-qz)^2+(pz-rx)^2+(qx-py)^2].$$

64. Show that the two tangent planes to $ax^2+by^2+cz^2=1$ which pass through the line given by

$$ny-mz=p,\quad lz-nx=q,\quad mx-ly=r,\quad lp+mq+nr=0;$$

are represented by the equation

$$\frac{(ny-mz-p)^2}{a}+\frac{(lz-nx-q)^2}{b}+\frac{(mx-ly-r)^2}{c}=(px+qy+rz)^2.$$

65. The tangent plane at a point P, not in a principal plane, and the normal at P to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ meet the plane XOY in a line QR and a point G. Show that QR is the polar of G with respect to the ellipse

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\quad z=0.$$

66. If the normals at points P and P′ on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

meet a principal plane in G and G′, show that the plane which bisects PP′ at right angles also bisects GG′.

67. If the normals at P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) on the ellipsoid

$x^2/a^2+y^2/b^2+z^2/c^2=1$ meet at the point (α, β, γ), show that PQ is given by the equations

$$a^2\left(\frac{\alpha x}{x_1x_2}-1\right)=b^2\left(\frac{\beta y}{y_1y_2}-1\right)=c^2\left(\frac{\gamma z}{z_1z_2}-1\right).$$

68. If the feet of three of the normals from (α, β, γ) to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ are (x_r, y_r, z_r), $r=1, 2, 3$, the equation of the plane through them can be written

$$\frac{a^4\alpha^2x}{(c^2-a^2)(a^2-b^2)x_1x_2x_3}+\frac{b^4\beta^2y}{(a^2-b^2)(b^2-c^2)y_1y_2y_3}$$
$$+\frac{c^4\gamma^2z}{(b^2-c^2)(c^2-a^2)z_1z_2z_3}+1=0.$$

69. Normals are drawn to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ at points on its section by the given plane $z=\gamma$. Show that the normals meet the plane XOY at points on the conic

$$\frac{a^2x^2}{(a^2-c^2)^2}+\frac{b^2y^2}{(b^2-c^2)^2}=1-\frac{\gamma^2}{c^2},\quad z=0\,;$$

and meet the planes YOZ, ZOX at points on the lines

$$x=0,\quad z=\gamma(1-a^2/c^2)\,;\quad y=0,\quad z=\gamma(1-b^2/c^2).$$

70. If the polar plane of P with respect to the conicoid

$$ax^2+by^2+cz^2=1$$

touches each of the conicoids

$$bx^2+cy^2+az^2=1,\quad cx^2+ay^2+bz^2=1,$$

show that P lies on the curve of intersection of the cone

$$(b-c)a^3x^2+(c-a)b^3y^2+(a-b)c^3z^2=0$$

and the surface

$$(b+c)a^3x^2+(c+a)b^3y^2+(a+b)c^3z^2=2abc.$$

71. The normal at a point P on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ meets the planes YOZ, ZOX in G and G′, and OQ is drawn from the origin O equal to and parallel to GG′. Show that the locus of Q is the ellipsoid $a^2x^2+b^2y^2+c^2z^2=(a^2-b^2)^2$.

72. The normal to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ at a point P on its curve of intersection with the plane $lx+my+nz=0$ is produced outwards to a point Q such that PQ . $p=k^2$, where p is the perpendicular from the centre to the tangent plane at P, and k is a constant. Show that Q lies on the conic

$$\frac{a^2lx}{a^2+k^2}+\frac{b^2my}{b^2+k^2}+\frac{c^2nz}{c^2+k^2}=0,\quad \frac{a^2x^2}{(a^2+k^2)^2}+\frac{b^2y^2}{(b^2+k^2)^2}+\frac{c^2z^2}{(c^2+k^2)^2}=1.$$

73. P is a point on the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ such that the planes through the centre and the generators through P are at right angles. Show that the normal at P intersects the conic

$$\frac{x^2}{a^2+c^2}+\frac{y^2}{b^2+c^2}=\frac{c^2}{a^2}+\frac{c^2}{b^2},\quad z=0.$$

74. If P (f, g, h), is a point on the cone $ax^2+by^2+cz^2=0$, three normals, other than the normal at P, pass through P. Show that the equation of the plane through the feet of these normals is

$$(a-b)(a-c)fx+(b-c)(b-a)gy+(c-a)(c-b)hz=bcf^2+cag^2+abh^2.$$

75. P is a point on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and ON is the perpendicular from the centre O to the tangent plane at P. If PN is of constant length K, show that ON lies on the cone

$$y^2z^2(b^2-c^2)^2+z^2x^2(c^2-a^2)^2+x^2y^2(a^2-b^2)^2$$
$$=\mathrm{K}^2(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2).$$

76. The feet of the normals to $x^2+2y^2+3z^2=984$ which lie in the plane $x+y+z=2$ are the points $(12, -18, 8)$, $(-6, 18, -10)$, and the normals intersect at the point $(-12, 54, -40)$.

77. The plane $x-z=4$ cuts the ellipsoid $x^2+2y^2+3z^2=50$ in a circle and the radii of the circle through the points $(6, \pm 1, 2)$ are the normals to the ellipsoid at these points.

78. If P, (x_1, y_1, z_1), is a point on the curve of intersection of the conicoid $ax^2+by^2+cz^2=1$ and its director sphere, show that the normal at P meets the conicoid again at the point Q given by

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{cz_1}=\frac{-2}{a+b+c},$$

and that the locus of Q is the curve of intersection of the conicoid and the cone

$$\frac{(b+c)a^2x^2}{(b+c-a)^2}+\frac{(c+a)b^2y^2}{(c+a-b)^2}+\frac{(a+b)c^2z^2}{(a+b-c)^2}=0.$$

79. The normals at (x_1, y_1, z_1), (x_2, y_2, z_2), on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

will intersect if

$$\frac{(b^2-c^2)x_1}{x_2-x_1}+\frac{(c^2-a^2)y_1}{y_2-y_1}+\frac{(a^2-b^2)z_1}{z_2-z_1}=0,$$

and if (α, β, γ) is the point of intersection,

$$a^2\alpha\left(\frac{1}{x_1}-\frac{1}{x_2}\right)=b^2\beta\left(\frac{1}{y_1}-\frac{1}{y_2}\right)=c^2\gamma\left(\frac{1}{z_1}-\frac{1}{z_2}\right).$$

80. The six normals from $(4/3, 2/3, 0)$ to the ellipsoid $x^2+2y^2+3z^2=24$ consist of the normals at $(2, 2, 2)$, $(2, 2, -2)$, the normal at $(4, -2, 0)$ counted twice, and the normals to the ellipse $x^2+2y^2=24$, $z=0$, at the points where the line $x-2y+4=0$, $z=0$ cuts it.

81. If the line $\dfrac{x-p}{l}=\dfrac{y-q}{m}=\dfrac{z-r}{n}$ is perpendicular to its polar with respect to the surface $ax^2+by^2+cz^2=1$, and meets the surface in P and Q, show that the normals at P and Q lie in the plane

$$bc(qn-rm)(x-p)+ca(rl-pn)(y-q)+ab(pm-ql)(z-r)=0,$$

and that PQ is a principal axis of the section of the surface by the plane

$$\begin{vmatrix} x-p, & y-q, & z-r \\ l, & m, & n \\ bc(qn-rm), & ca(rl-pn), & ab(pm-ql) \end{vmatrix}=0.$$

82. Show that the points $(1, -2, 2)$, $(-5, 4, 8)$ are the extremities of a normal chord of the hyperboloid $2x^2 + y^2 - z^2 = 2$ and that the chord is a principal axis of the section of the surface by the plane

$$4x - y + 5z = 16.$$

83. P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2), R, (x_3, y_3, z_3) are the extremities of three conjugate diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Show that $(x - x_1)(x_2 - x_3) + (y - y_1)(y_2 - y_3) + (z - z_1)(z_2 - z_3) = 0$ represents the plane drawn through the normal at P at right angles to the plane PQR and deduce that the projections on the plane PQR of the normals at P, Q, R are concurrent at the orthocentre of the triangle PQR.

84. OP, OQ, OR are three conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. OP, OQ are produced to P′ and Q′ so that OP′/OP = OQ′/OQ = k, where k is a constant. Show that the enveloping cones whose vertices are P′ and Q′ intersect in two planes, of which one is the plane through OR and the mid-point of PQ, and the other touches the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2}{k^2}.$$

85. OP, OQ, OR are conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. OP′, OQ′, OR′ are drawn at right angles to the planes QOR, ROP, POQ, and of lengths such that kOP′, kOQ′, kOR′ are equal to the areas of the triangles QOR, ROP, POQ, respectively, k being a certain fixed length. Show that OP′, OQ′, OR′ are conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{b^2c^2} + \frac{y^2}{c^2a^2} + \frac{z^2}{a^2b^2} = \frac{1}{4k^2}.$$

86. OP, OQ, OR are conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Show that the sides of the triangle PQR touch the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = \frac{1}{2}$ at their mid-points. Show also that the necessary and sufficient condition that the line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

should cut the ellipsoid in the ends of two conjugate diameters is

$$2a^2(m\gamma - n\beta)^2 + 2b^2(n\alpha - l\gamma)^2 + 2c^2(l\beta - m\alpha)^2 = b^2c^2l^2 + c^2a^2m^2 + a^2b^2n^2.$$

87. The equations of the axes of the conic $3x^2 + y^2 - z^2 = 3$, $2x + y + 2z = 0$ are $x/1 = y/2 = z/-2$, $x/2 = y/-2 = z/-1$, and the lengths of the axes are 6 and $6/\sqrt{5}$. The axes are the lines in which the given plane cuts the cone $1/x - 1/y + 1/z = 0$. The axes of all sections parallel to the given plane lie in the planes

$$6x - 2y + z = 0, \quad 3x + 2y + 2z = 0.$$

88. The lengths of the axes of the conic

$$-x^2 + 20y^2 + 6z^2 = 8, \quad 3x + 6y + 2z = 0,$$

are 4, 2; the equations of the axes are $x/2 = y/-3 = z/6$, $x/-6 = y/2 = z/3$; the equations of the equi-conjugate diameters of the conic are

$$x/2 = y/-1 = z/0, \quad x/-10 = y/1 = z/12.$$

89. The foci of the conic $x^2/9+y^2/4+z^2/1=21$, $2x+3y-6z=0$, are $(9,\ -4,\ 1)$, $(-9,\ 4,\ -1)$.

90. The centre of the conic $14x^2+5y^2-4z^2=72$, $2x-y-2z+36=0$, is $(10,\ -14,\ 35)$; the lengths of the semi-axes are 18, 36; the extremities of the axes are $(22,\ -2,\ 41)$, $(-2,\ -26,\ 29)$; $(22,\ -38,\ 59)$, $(-2,\ 10,\ 11)$.

91. The centre of the conic $2x^2+y^2=12z$, $6x-3y-6z=7$ is $(3,\ -3,\ 10/3)$; the lengths of the semi-axes are $(39/2)^{\frac{1}{2}}$, $(39/4)^{\frac{1}{2}}$, and the directions of the axes are given by $2:2:1$, $1:-2:2$.

92. The centre of the conic $5x^2+17y^2-7z^2=0$, $x-2y+2z=81$, is $(-119,\ 70,\ 170)$; the lengths of the semi-axes are $3(595)^{\frac{1}{2}}$, $9(595)^{\frac{1}{2}}$; their directions are given by $2:-1:-2$, $2:2:1$.

93. The axes of the conic $xy=az$, $lx+my+nz=0$, are given by

$$\frac{nx+am}{(m^2+n^2)^{\frac{1}{2}}}=\frac{ny+al}{\pm(n^2+l^2)^{\frac{1}{2}}}=\frac{n^2z-2alm}{-l(m^2+n^2)^{\frac{1}{2}}\mp m(n^2+l^2)^{\frac{1}{2}}}.$$

94. If the plane $lx+my+nz=0$ cuts the paraboloid $xy=az$ in a hyperbola of eccentricity 2, the line $x/l=y/m=z/n$ lies on the cone $z^2(x^2+y^2+z^2)=3x^2y^2$.

95. Show that the director circles of sections of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, whose planes are parallel to the fixed plane $lx+my+nz=0$, lie on the conicoid

$$\begin{aligned}(lx+my+nz)&[(b^2+c^2-a^2)lx+(c^2+a^2-b^2)my+(a^2+b^2-c^2)nz]\\&=-(a^2l^2+b^2m^2+c^2n^2)\\&\quad\times\left(x^2+y^2+z^2-\frac{a^2l^2(b^2+c^2)+b^2m^2(c^2+a^2)+c^2n^2(a^2+b^2)}{a^2l^2+b^2m^2+c^2n^2}\right).\end{aligned}$$

96. Show that if $a^2+b^2=2c^2$, the generators through any one of the ends of an equi-conjugate diameter of the principal elliptic section of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ are at right angles.

97. Show that $x=1+\lambda y=\dfrac{2z}{\lambda}-1$ represents a generator of the hyperboloid $x^2-2yz=1$, and that any point on the surface is given by

$$x=\frac{\mu+\lambda}{\mu-\lambda},\quad y=\frac{2}{\mu-\lambda},\quad z=\frac{\mu\lambda}{\mu-\lambda}.$$

Show also that if the generators through P are at right angles, P lies on the circle in which the plane $y+z=0$ cuts the surface.

98. Show that the section of the hyperboloid $4x^2+y^2-2z^2=12$ whose centre is $(-3,\ 0,\ 6)$ is circular and that its area is twice that of the parallel central section. Show that the sphere which has the given circular section as a great circle is

$$(x-1)(x+7)+(y-4)(y+4)+(z-2)(z-10)=0.$$

99. The radius of a real circular cylinder which circumscribes the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is b and the area of its curve of contact with the ellipsoid is $\pi(b^2c^2-c^2a^2+a^2b^2)^{\frac{1}{2}}$, $(a>b>c)$.

100. If each of two tangent cones to a conicoid has sets of three mutually perpendicular generators, their common points lie in two perpendicular planes.

101. Two circular sections of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ of opposite systems have the same radius r. Show that the square of the radius of the sphere through them is

$$b^2-\frac{a^2-c^2}{a^2}(b^2-r^2), \quad \text{or} \quad b^2+\frac{a^2-c^2}{c^2}(b^2-r^2), \quad (a>b>c).$$

102. P, Q, R, S are vertices of a skew quadrilateral whose sides lie along generators of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$. Show that $(z_P+z_R)(z_Qz_S-c^2)=(z_Q+z_S)(z_Pz_R-c^2)$.

103. P, (x', y', z'), is a point on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and ON is the perpendicular from the centre to the tangent plane at P. If OP $=r$ and ON $=p$, show that the bisector of the angle PON is given by

$$\frac{a^2x}{x'(pr+a^2)}=\frac{b^2y}{y'(pr+b^2)}=\frac{c^2z}{z'(pr+c^2)},$$

and that, if a point Q is taken on the bisector such that OQ^2=ON . OP, Q lies on the surface

$$\frac{x^2}{a^2+x^2+y^2+z^2}+\frac{y^2}{b^2+x^2+y^2+z^2}+\frac{z^2}{c^2+x^2+y^2+z^2}=\frac{1}{2}.$$

104. If the plane $lx+nz=p$ cuts the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

in a parabola, show that $a^2l^2=c^2n^2$ and that the vertex of the parabola is the point

$$\left(\frac{p^2+a^2l^2}{2pl},\ 0,\ \frac{p^2-c^2n^2}{2pn}\right).$$

105. If $a+b+c=0$, the section of the cone $ax^2+by^2+cz^2=0$ by the plane $ax+by+cz=18abc$ is a parabola whose latus rectum is of length $4\sqrt{3}(a^2+b^2+c^2)$; the axis is parallel to the line $x=y=z$; the vertex is $(a^2+5bc,\ b^2+5ca,\ c^2+5ab)$; and the focus is

$$[3(a^2+bc),\ 3(b^2+ca),\ 3(c^2+ab)].$$

106. If the plane $lx+my+nz+p=0$ cuts the cone

$$x^2/a+y^2/b+z^2/c=0$$

in a parabola, the directrix lies in the plane

$$(b+c-a)lx+(c+a-b)my+(a+b-c)nz=0,$$

and the focus is given by

$$\frac{x}{a(b+c-a)l}=\frac{y}{b(c+a-b)m}=\frac{z}{c(a+b-c)n}=\frac{p}{2(a^2l^2+b^2m^2+c^2n^2)}.$$

107. The hyperbolic paraboloid which has the lines

$$a_rx+b_ry=1, \quad z=c_r, \quad r=1, 2, 3,$$

as generators is given by

$$\begin{vmatrix} \dfrac{a_1x+b_1y-1}{z-c_1}, & a_1, & b_1 \\ \dfrac{a_2x+b_2y-1}{z-c_2}, & a_2, & b_2 \\ \dfrac{a_3x+b_3y-1}{z-c_3}, & a_3, & b_3 \end{vmatrix} = 0.$$

108. The equation of the hyperboloid which has the coordinate axes for principal axes and the line $x-a=y-b=z-c$ as a generator is

$$\frac{x^2}{(c-a)(a-b)}+\frac{y^2}{(a-b)(b-c)}+\frac{z^2}{(b-c)(c-a)}+1=0,$$

and the equations of the other generator through the point (a, b, c) are

$$\frac{x-a}{-bc+ca+ab}=\frac{y-b}{bc-ca+ab}=\frac{z-c}{bc+ca-ab}.$$

109. The vertices of a tetrahedron are (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$. Show that the altitudes are generators of the hyperboloid

$$a^2x^2(b^2-c^2)+b^2y^2(c^2-a^2)+c^2z^2(a^2-b^2)+(b^2-c^2)(c^2-a^2)(a^2-b^2)=0.$$

110. Show that the equations of the generators of the hyperboloid

$$8x^2+8y^2-4z^2-8yz-8zx+4xy=3$$

which are at right angles to the planes of the circular sections are

$$2x=2y\pm 2=2z\pm 1, \quad 10x=10y\pm 10=-2z\pm 5.$$

111. The condition that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$ should have generators at right angles to the planes of its circular sections is

$$\begin{vmatrix} a-b-c, & 2h, & 2g \\ 2h, & b-c-a, & 2f \\ 2g, & 2f, & c-a-b \end{vmatrix} = 0.$$

112. A variable circle with its centre on OZ and its plane at right angles to the line $x=a$, $y=z$, intersects that line. Show that it generates the hyperboloid $2x^2+3y^2-z^2-2yz=2a^2$. Show that the same surface is generated by a variable circle with its centre on the line $x=0$, $2y=z$, which has its plane at right angles to the line $x=a$, $3y+z=0$, and intersects that line.

113. PP′ is a diameter of the principal elliptic section of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$. A variable generator intersects a generator through P and the generator of the same system through P′ in the points Q and Q′. Show that $z_Q z_{Q'}=-c^2$.

114. Show that the two generators through the point (α, β, γ) on the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ are the lines in which the tangent plane at the point meets the planes

$$\frac{(\beta x-\alpha y)^2}{a^2b^2}=\frac{(z-\gamma)^2}{c^2}.$$

Show that the same result holds for the paraboloid $x^2/a^2-y^2/b^2=2z/c$.

115. Show that the equations of the generators of $yz+zx+xy+a^2=0$ are

$$y-a=\lambda(x+a),\ \lambda(z-a)+y+z=0;\ z-a=\mu(x+a),\ \mu(y-a)+y+z=0;$$

and that any point on the surface is given by

$$\frac{x}{2+\lambda+\mu+\lambda\mu}=\frac{y}{\lambda-\mu-\lambda\mu}=\frac{z}{-\lambda+\mu-\lambda\mu}=\frac{-a}{\lambda+\mu+\lambda\mu}.$$

Show that

$$y=a,\ z=-a;\quad z=a,\ x=-a;\quad x=a,\ y=-a;\quad x-a=-2y=z+a,$$

are generators of the same system and that, if they meet any generator of the opposite system in P, Q, R, S, respectively, (PR, QS) is a harmonic range.

116. Establish the following results involving the coordinates of a line (see Appendix, Note to § 43).

(1) If the line $(l, m, n, \lambda, \mu, \nu)$ lies in the plane $ax+by+cz+d=0$,

$$al+bm+cn=0 \quad\text{and}\quad \frac{b\nu-c\mu}{l}=\frac{c\lambda-a\nu}{m}=\frac{a\mu-b\lambda}{n}=d.$$

(2) If the lines $(l, m, n, \lambda, \mu, \nu)$, $(l', m', n', \lambda', \mu', \nu')$ intersect,

$$l\lambda'+m\mu'+n\nu'+l'\lambda+m'\mu+n'\nu=0,$$

and the plane through the lines is given by

$$l'(ny-mz-\lambda)+m'(lz-nx-\mu)+n'(mx-ly-\nu)=0,$$

or by $l(n'y-m'z-\lambda')+m(l'z-n'x-\mu')+n(m'x-l'y-\nu')=0.$

(3) The line $(l, m, n, \lambda, \mu, \nu)$ touches the surface

$$ax^2+by^2+cz^2=1$$

if

$$bc\lambda^2+ca\mu^2+ab\nu^2=al^2+bm^2+cn^2.$$

(4) The tangent planes to $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose line of intersection has coordinates $l, m, n, \lambda, \mu, \nu$, are given by

$$a^2(ny-mz-\lambda)^2+b^2(lz-nx-\mu)^2+c^2(mx-ly-\nu)^2=(\lambda x+\mu y+\nu z)^2,$$

and, if the axes are rectangular and the planes are perpendicular,

$$l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)=\lambda^2+\mu^2+\nu^2.$$

(5) The tangent planes to $x^2/a+y^2/b=2z$ whose line of intersection has coordinates $l, m, n, \lambda, \mu, \nu$, are given by

$$a(ny-mz-\lambda)^2+b(lz-nx-\mu)^2=2(mx-ly-\nu)(\lambda x+\mu y+\nu z).$$

(6) The normals at the points where the line $(l, m, n, \lambda, \mu, \nu)$ meets the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ intersect if $a^2l\lambda+b^2m\mu+c^2n\nu=0$, and the equation of the plane containing them is

$$a^2(b^2-c^2)mnx+b^2(c^2-a^2)nly+c^2(a^2-b^2)lmz=b^2c^2l\lambda+c^2a^2m\mu+a^2b^2n\nu.$$

(7) The line $(l, m, n, \lambda, \mu, \nu)$ is a diameter of the section of the surface $ax^2+by^2+cz^2=1$ by the plane

$$\frac{\lambda}{a}(ny-mz-\lambda)+\frac{\mu}{b}(lz-nx-\mu)+\frac{\nu}{c}(mx-ly-\nu)=0.$$

(8) The polar line of $(l, m, n, \lambda, \mu, \nu)$ with respect to the surface $ax^2+by^2+cz^2=1$ is $(bc\lambda,\ ca\mu,\ ab\nu,\ -al,\ -bm,\ -cn)$, and if the line is a generator of the surface,

$$\frac{al}{\lambda}=\frac{bm}{\mu}=\frac{cn}{\nu}=-\frac{lmn}{\lambda\mu\nu}=\pm\sqrt{-abc}.$$

(9) If the line $(l, m, n, \lambda, \mu, \nu)$ is a generator of the surface

$$x^2/a^2-y^2/b^2=2z/c,$$

$$l/a=\pm m/b,\quad a\lambda=\pm b\mu,\quad c\nu=\pm abn\ ;$$

if it is a generator of $ayz+bzx+cxy+abc=0$,

$$\frac{\lambda l^2}{a}=\frac{\mu m^2}{b}=\frac{\nu n^2}{c}=\pm lmn.$$

(10) The conicoid which has the coordinate axes as conjugate diameters and the line $(l, m, n, \lambda, \mu, \nu)$ as a generator is given by

$$\lambda mnx^2+\mu nly^2+\nu lmz^2+\lambda\mu\nu=0.$$

MISCELLANEOUS EXAMPLES II.

1. Prove that the lines

$$\frac{x+1}{2}=\frac{y+1}{3}=\frac{z+1}{4}, \quad x+2y+3z-14=0=3x+4y+5z-26$$

are coplanar, and find the coordinates of their point of intersection.

Ans. (1, 2, 3).

2. Show that the planes

$$x-y+z-4=0, \quad 2x-y-z+4=0, \quad x+y-5z+14=0$$

form a triangular prism, and calculate the breadth of each face of the prism.

Ans. $\sqrt{\frac{27}{14}}, \sqrt{\frac{27}{14}}, \sqrt{\frac{12}{7}}$.

3. Find the angle between the lines

$$\frac{x-7}{2}=\frac{y+3}{-1}=\frac{z-4}{1}; \quad 6x+4y-5z=4, \ x-5y+2z=12;$$

show that the lines intersect, and find the equation of the plane containing them.

Ans. $\pi/3$, $x+y-z=0$.

4. Find the equations of the line drawn through the point (3, −4, 1) parallel to the plane $2x+y-z=5$, so as to intersect the line

$$\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-2}{-1}.$$

Find also the coordinates of the point of intersection, and the equation of the plane through the given line and the required line.

Ans. $\frac{x-3}{-1}=\frac{y+4}{3}=\frac{z-1}{1}$, (1, 2, 3), $y-3z+7=0$.

5. Prove that the equations of the perpendicular from the origin to the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ are

$$x(bc'-b'c)+y(ca'-c'a)+z(ab'-a'b)=0,$$
$$x(ad'-a'd)+y(bd'-b'd)+z(cd'-c'd)=0.$$

6. Prove that the planes

$$6x+y+5z+5=0, \quad 2x+7y+3z+3=0$$

intersect the line joining the points (1, 2, 3), (3, −5, 1) in points which form a harmonic range with the given points.

7. Obtain, in the symmetrical form, the equations of the projection of the line

$$8x-7y-z+6=0=x+y+z-3$$

on the plane $5x-y-4z=3$.

Ans. $\dfrac{x-1}{1}=\dfrac{y-2}{-3}=\dfrac{z}{2}$.

8. Prove that $4x^2+8y^2+z^2-6yz+5zx-12xy=0$ represents a pair of planes, and find the angle between them. Find also the angle between the lines in which the plane $11x-13y+2z=0$ cuts them.

Ans. $\cos^{-1}\dfrac{13}{3\sqrt{22}}$, $90°$.

9. Find the length of the shortest distance between the lines

$$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4},\quad 2x+3y-5z-6=0=3x-2y-z+3.$$

Ans. $97/13\sqrt{6}$.

10. Find the equations of the straight line perpendicular to both of the lines

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3},\quad \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2},$$

and passing through their intersection.

Ans. $\dfrac{x-2}{7}=\dfrac{y-3}{4}=\dfrac{z-1}{-5}$.

11. For the lines

$$\frac{x-2}{0}=\frac{y-1}{1}=\frac{z}{1},\quad \frac{x-3}{2}=\frac{y-5}{2}=\frac{z+1}{1},$$

find the length and equations of the shortest distance and the co-ordinates of the points where it meets the given lines.

Ans. 3, $\dfrac{x-2}{1}=\dfrac{y-1}{-2}=\dfrac{z}{2}$, $(2, 1, 0)$, $(1, 3, -2)$.

12. A square **ABCD**, of diagonal $2a$, is folded along the diagonal **AC**, so that the planes **DAC**, **BAC** are at right angles. Show that the shortest distance between **DC** and **AB** is then $2a/\sqrt{3}$.

13. Find the equation of the plane which passes through the line $3x+5y+7z-5=0=x+y+z-3$, and is parallel to the line

$$4x+y+z=0=2x-3y-5z.$$

Ans. $x+2y+3z=1$.

14. Find the equation of the right circular cylinder whose axis is $x=2y=-z$ and radius 4, and prove that the area of the section of the cylinder by the plane **XOY** is 24π

Ans. $5x^2+8y^2+5z^2+4yz+8zx-4xy=144$.

15. Find the equation of the right circular cone which passes through the line $2x=3y=-5z$, and has the line $x=y=z$ as axis.

Ans. $yz+zx+xy=0$.

16. Obtain the formulae for the transformation from a set of rectangular axes **OX**, **OY**, **OZ**, to a second rectangular set, **O**ξ, **O**η, **O**ζ, where **O**ξ lies in the plane **XOY** and makes an angle θ with **OX** and **O**ζ makes an angle ϕ with **OZ**, viz.

$$x=\xi\cos\theta-\eta\sin\theta\cos\phi+\zeta\sin\theta\sin\phi,$$
$$y=\xi\sin\theta+\eta\cos\theta\cos\phi-\zeta\cos\theta\sin\phi,$$
$$z=\eta\sin\phi+\zeta\cos\phi.$$

Apply these to show that the section of the surface

$$3x^2-y^2+z^2+yz+zx=4$$

by the plane $2x-2y+z=0$ is a circle of radius 2.

17. If l_r, m_r, n_r $(r=1, 2, 3)$ are the direction-cosines of three mutually perpendicular lines, prove that

$$m_2n_2(m_3{}^2-n_3{}^2)-m_3n_3(m_2{}^2-n_2{}^2)=l_1l_2l_3$$

and that

$$l_1m_1n_1(m_2n_3+m_3n_2)+l_2m_2n_2(m_3n_1+m_1n_3)+l_3m_3n_3(m_1n_2+m_2n_1)=l_1l_2l_3.$$

18. If **O**ξ, **O**η, **O**ζ are a second set of rectangular axes whose direction-cosines referred to **OX**, **OY**, **OZ** are l_r, m_r, n_r $(r=1, 2, 3)$ and the projections of **O**ξ and **O**η on the plane **XOY** make angles ϕ_1 and ϕ_2 with **OX**, prove that

$$\tan(\phi_1-\phi_2)=\pm\frac{n_3}{n_1n_2}.$$

19. Find the surface generated by a variable line which intersects the parabolas $x=2am$, $y=0$, $z=am^2$; $x=0$, $y=2an$, $z=-an^2$, and is parallel to the plane $x+y=0$.

Ans. $x^2-y^2=4az$.

20. A line **PQ** moves with its extremities on the lines

$$y=mx,\ z=c;\quad y=-mx,\ z=-c,$$

so as to subtend a right angle at the origin. Prove that the foot of the perpendicular from the origin to **PQ** lies on the curve of intersection of the surfaces

$$(y^2-m^2x^2)(1-m^2)=(z^2-c^2)m^2 \text{ and } mz(x^2+y^2+z^2-c^2)=cxy(1+m^2).$$

21. Lines are drawn parallel to the plane $x=0$ to intersect the parabola $y^2=4ax$, $z=0$; and the line $y=0$, $z=c$. Prove that they generate the surface $c^2y^2=4ax(z-c)^2$.

22. Find the surface generated by a straight line which intersects the lines $y=0$, $z=c$; $x=0$, $z=-c$

and the hyperbola $z=0$, $xy+c^2=0$.

Ans. $z^2-xy=c^2$.

23. Prove that lines drawn from the origin so as to touch the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

lie on the cone $d(x^2+y^2+z^2)=(ux+vy+wz)^2$.

24. Find the equations of the spheres which pass through the circle

$$x^2+y^2+z^2=1,\quad 2x+4y+5z=6$$

and touch the plane $z=0$.

Ans. $x^2+y^2+z^2-2x-4y-5z+5=0$,
$5x^2+5y^2+5z^2-2x-4y-5z+1=0$.

25. Prove that the locus of the centres of spheres which touch the two lines

$$y = \pm mx, \quad z = \pm c$$

is the surface $xym + cz(1+m^2) = 0$.

26. P and Q are points on the lines

$$y = mx,\ z = c\,; \quad y = -mx,\ z = -c$$

such that PQ is always parallel to the plane ZOX. Prove that the circle on PQ as diameter whose plane is parallel to the plane ZOX generates the surface

$$m^2x^2 - y^2 + m^2z^2 = m^2c^2.$$

27. Prove that the plane

$$(x-\alpha)(u+\alpha) + (y-\beta)(v+\beta) + (z-\gamma)(w+\gamma) = 0$$

cuts the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

in a circle whose centre is (α, β, γ), and that the equation of the sphere which has this circle for a central section is

$$x^2+y^2+z^2-2\alpha(x-u)-2\beta(y-v)-2\gamma(z-w)+2\alpha^2+2\beta^2+2\gamma^2+d=0.$$

28. If three mutually perpendicular lines whose direction-cosines are l_r, m_r, n_r $(r=1, 2, 3)$ are drawn from the origin O to meet the sphere $x^2+y^2+z^2=a^2$ in P, Q, R, prove that the equation of the plane PQR is

$$(l_1+l_2+l_3)x + (m_1+m_2+m_3)y + (n_1+n_2+n_3)z = a,$$

and that the radius of the circle PQR is $\sqrt{(2/3)}a$.

29. If A, B, C are the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, and the axes are rectangular, prove that the diameter of the circle ABC is

$$\sqrt{\frac{(b^2+c^2)(c^2+a^2)(a^2+b^2)}{b^2c^2+c^2a^2+a^2b^2}}.$$

30. Find the radius of the circle

$$2x - y - 2z + 13 = 0, \quad x^2+y^2+z^2 = 2x+4y+4z+1,$$

and the equation of the right circular cylinder which has the circle for a normal section.

Ans. 1, $5x^2+8y^2+5z^2-4yz+8zx+4xy-34x-28y-20z+56=0$.

31. Show that the plane $7x+4y+z=0$ cuts the cones

$$35x^2-8y^2-3z^2=0, \quad 7yz+8zx-3xy=0$$

in the same pair of perpendicular lines, and that the equation of the plane through the other two common generators of the cones is

$$5x-2y-3z=0.$$

32. An ellipse whose axes are of lengths $2a$ and $2b$ $(a>b)$ moves with its major axis parallel to OX and its minor axis parallel to OY, so that three mutually perpendicular lines can be drawn from O to intersect it. Prove that its centre lies on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1.$$

33. Two right circular cones have a common vertex and axis, and their semivertical angles are $\pi/4$ and $\pi/3$ respectively. Show that any tangent plane to the first cuts the second in perpendicular generators.

34. **OP, OQ, OR**, whose direction-cosines are l_r, m_r, n_r $(r=1, 2, 3)$, are three mutually perpendicular generators of the cone

$$ax^2+by^2+cz^2=0, \quad (a+b+c=0).$$

Prove that the three planes through **OP, OQ, OR**, at right angles respectively to the tangent planes which touch along **OP, OQ, OR**, pass through the line

$$\frac{(b-c)x}{l_1l_2l_3}=\frac{(c-a)y}{m_1m_2m_3}=\frac{(a-b)z}{n_1n_2n_3}.$$

35. If **P** is any point on the curve of intersection of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the cone $\frac{x^2}{a^4}+\frac{y^2}{b^4}+z^2\left(\frac{1}{c^4}-\frac{1}{k^4}\right)=0$, the tangent plane at **P** to the ellipsoid cuts the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in an ellipse of constant area.

36. If the axes are rectangular, find the length of the chord of the conicoid $x^2+4y^2-4z^2=28$, which is parallel to the plane $3x-2y+2z=0$, and is bisected at the point $(2, 1, -2)$.

Ans. $\sqrt{154}$.

37. Find the equation of the tangent plane to the surface

$$3x^2+2y^2-6z^2=6,$$

which passes through the point $(3, 4, -3)$, and is parallel to the line

$$x=y=-z.$$

Ans. $x+y+2z=1$.

38. Prove that for all values of λ the plane

$$\frac{2x}{a}+\frac{y}{b}+\frac{2z}{c}-1+\lambda\left(\frac{x}{a}-\frac{2y}{b}-\frac{z}{c}-2\right)=0$$

is a tangent plane to the conicoid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

and that its point of contact lies on the line

$$\frac{4x-5a}{3a}=\frac{y}{b}=\frac{4z+3c}{-5c}.$$

39. Prove that the straight lines joining the origin to the points of contact of a common tangent plane to the conicoids

$$ax^2+by^2+cz^2=1, \quad (a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2=1,$$

are at right angles.

40. Find the equations of the two planes that can be drawn through the line $x=4$, $3y+4z=0$ to touch the conicoid $x^2+3y^2-6z^2=4$.

Ans. $x+9y+12z=4$, $x-9y-12z=4$.

41. Find the points of contact of the tangent planes to the conicoid $2x^2-25y^2+2z^2=1$, which intersect in the line joining the points $(-12, 1, 12)$, $(13, -1, -13)$.

Ans. $(3, -1, 2)$, $(-2, -1, -3)$.

42. Find the equations of the two plane sections of the surface $x^2-y^3+2z^2=1$, which have their centres on the line

$$\frac{x-4}{3}=\frac{y-3}{1}=\frac{z-7}{4},$$

and pass through the point $(-1, -5, 1)$.

Ans. $x-2y+6z-15=0,\ 2x+y+2z+5=0.$

43. **M**, the mid-point of the normal chord at **P**, a point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

lies on the plane **ZOX**. Prove that **P** must lie on one of the planes

$$\frac{\sqrt{a^2-b^2}}{a^3}x\pm\frac{\sqrt{b^2-c^2}}{c^3}z=0,$$

and **M** on one of the lines

$$\frac{x}{a\sqrt{a^2-b^2}}=\frac{y}{0}=\pm\frac{z}{c\sqrt{b^2-c^2}}.$$

44. **P** is any point on the curve of intersection of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, and the normal at **P** meets the plane **XOY** in **G**. From **O** (the origin) **OQ** is drawn equal and parallel to **GP**. Prove that **Q** lies on the conic

$$ax+by+cz=c^2,\quad a^2x^2+b^2y^2+c^2z^2=c^4.$$

45. A cone is described whose vertex is **A**, and whose base is the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $z=0$. It meets the given conicoid at points in the plane **XOY**, and at points in a plane **Q**. Prove that if **A** lies in a fixed plane **P**, the pole of **Q**, with respect to the given conicoid, lies on a conicoid which touches the given conicoid at all points of its intersection with the plane **P**.

46. If the polar planes of **P** and **Q** with respect to a central conicoid meet in a line **AB**, and the plane through **Q** parallel to the polar plane of **P** meets the plane through **P** parallel to the polar plane of **Q** in a line **CD**, show that the plane through **AB** and **CD** passes through the centre of the conicoid.

47. Prove that the equation of the cone whose vertex is (α, β, γ) and base the conic $ax^2+by^2+cz^2=1$, $\mathrm{P}\equiv lx+my+nz-p=0$ is obtained by eliminating λ from the equations

$$a(\lambda x+\alpha)^2+b(\lambda y+\beta)^2+c(\lambda z+\gamma)^2=(\lambda+1)^2$$

and $\lambda\mathrm{P}+\mathrm{P}'=0$, where $\mathrm{P}'\equiv l\alpha+m\beta+n\gamma-p$, and then obtain the equation of the cone in the form

$$\mathrm{P}^2(a\alpha^2+b\beta^2+c\gamma^2-1)-2\mathrm{PP}'(a\alpha x+b\beta y+c\gamma z-1)$$
$$+\mathrm{P}'^2(ax^2+by^2+cz^2-1)=0.$$

Deduce (i) the equation of the cone whose vertex is (α, β, γ) and base $ax^2+by^2=1$, $z=0$, and (ii) the equation of the other plane section of the conicoid and the cone.

48. Prove that the locus of centres of sections of $ax^2+by^2+cz^2=1$ which pass through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is a conic whose centre is given by $\frac{2x-\alpha}{l}=\frac{2y-\beta}{m}=\frac{2z-\gamma}{n}=-\frac{a\alpha l+b\beta m+c\gamma n}{al^2+bm^2+cn^2}$.

49. If the axes are rectangular and **P**, (x_1, y_1, z_1), **Q**, (x_2, y_2, z_2), **R**, (x_3, y_3, z_3) are the extremities of three conjugate diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the planes through the centre and the normals at **P**, **Q**, **R** pass through the line

$$\frac{a^4(b^2-c^2)x}{x_1x_2x_3}=\frac{b^4(c^2-a^2)y}{y_1y_2y_3}=\frac{c^4(a^2-b^2)z}{z_1z_2z_3}.$$

50. If **OP**, **OQ**, **OR** are conjugate semidiameters of the ellipsoid whose equation referred to rectangular axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the areas of the triangles **QOR**, **ROP**, **POQ** are equal, the planes **QOR**, **ROP**, **POQ** touch the cone

$$\frac{x^2}{a^2(a^2b^2+a^2c^2-2b^2c^2)}+\frac{y^2}{b^2(b^2c^2+b^2a^2-2c^2a^2)}+\frac{z^2}{c^2(c^2a^2+c^2b^2-2a^2b^2)}=0.$$

51. If **P**(x_1, y_1, z_1), **Q**(x_2, y_2, z_2), **R**(x_3, y_3, z_3) are the extremities of three conjugate semidiameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the cone through the coordinate axes and **OP**, **OQ**, **OR** is given by $\frac{x_1x_2x_3}{a^2x}+\frac{y_1y_2y_3}{b^2y}+\frac{z_1z_2z_3}{c^2z}=0$, and the cone which touches the coordinate planes and the planes **QOR**, **ROP**, **POQ** by

$$\frac{\sqrt{xx_1x_2x_3}}{a^2}+\frac{\sqrt{yy_1y_2y_3}}{b^2}+\frac{\sqrt{zz_1z_2z_3}}{c^2}=0.$$

52. If one axis of a central section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lies in the plane $ux+vy+wz=0$, the other lies on the cone

$$\frac{a^2(b^2-c^2)u}{x}+\frac{b^2(c^2-a^2)v}{y}+\frac{c^2(a^2-b^2)w}{z}=0.$$

53. If the radius of the director circle of the conic $lx+my+nz=0$, $x^2/a^2+y^2/b^2+z^2/c^2=1$ is of constant length r, prove that the plane of the conic touches the cone

$$\frac{x^2}{a^2(b^2+c^2-r^2)}+\frac{y^2}{b^2(c^2+a^2-r^2)}+\frac{z^2}{c^2(a^2+b^2-r^2)}=0.$$

54. Prove that the equal conjugate diameters of the conic in which the plane $x+y+z=0$ cuts the conicoid $2x^2+3y^2+4z^2=1$ are the lines in which the plane cuts the cone $8x^2-y^2-10z^2=0$.

55. If a plane section of $ax^2+by^2+cz^2=1$ has one of its axes along the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ its equation must be

$$(\lambda^2+\mu^2+\nu^2)(a\lambda x+b\mu y+c\nu z)=(a\lambda^2+b\mu^2+c\nu^2)(\lambda x+\mu y+\nu z).$$

56. Prove that the lengths of the semi-axes of the section of the surface $2yz+2zx+2xy+a^2=0$ by the plane $lx+my+nz=0$ are

$$a \quad\text{and}\quad a(l^2+m^2+n^2)^{\frac{1}{2}}(2mn+2nl+2lm-l^2-m^2-n^2)^{-\frac{1}{2}}.$$

By considering the cases (i) $l+m+n=0$, (ii) $l=m=n$, show that the surface is a hyperboloid of revolution whose axis is $x=y=z$ and whose equation referred to its principal axes is $x^2+y^2-2z^2=a^2$.

57. Prove that the axes of the conic in which the plane

$$lx+my+nz=0$$

cuts the paraboloid $ax^2+by^2=2z$ are the lines in which the plane cuts the cone

$$\frac{l}{anx+l}-\frac{m}{bny+m}+\frac{(a-b)n^2}{abn^2z-bl^2-am^2}=0.$$

58. The fixed plane $lx+my+nz=p$, where $\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=0$, cuts the cone $ax^2+by^2+cz^2=0$ in a parabola. Prove that the axes of all parallel parabolic sections lie in the plane

$$\frac{a^2(b-c)x}{l}+\frac{b^2(c-a)y}{m}+\frac{c^2(a-b)z}{n}=0,$$

and the vertices on the line

$$\frac{lx}{\frac{b^3}{m^2}(c-a)^2+\frac{c^3}{n^2}(a-b)^2}=\frac{my}{\frac{c^3}{n^2}(a-b)^2+\frac{a^3}{l^2}(b-c)^2}=\frac{nz}{\frac{a^3}{l^2}(b-c)^2+\frac{b^3}{m^2}(c-a)^2}.$$

59. If the axes are rectangular and a line moves so as to intersect the lines $x=a$, $y=0$; $x=-a$, $y=mz$ and the circle $x^2+y^2=a^2$, $z=0$, prove that its locus is a hyperboloid of one sheet whose circular sections are perpendicular to the given lines.

60. Prove that the centre of any sphere which passes through the origin and through a circular section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad (a>b>c),$$

lies on the hyperbola

$$y=0, \quad \frac{a^2x^2}{a^2-b^2}-\frac{c^2z^2}{b^2-c^2}=b^2$$

61. Prove that the cylinders

$$41x^2-24xy+34y^2=25, \quad 25x^2+40xz+34z^2=9$$

have a common circular section, finding its equation and radius.

Ans. $4x-3y+5z=0$, 1.

62. Prove that any enveloping cone of the conicoid $x^2+4yz-2z^2=a^2$, whose vertex lies on the z-axis, touches the conicoid at all points of a circle.

63. **O** is an umbilic of an ellipsoid and **OP** is the normal chord through **O**. The tangent planes at **O** and **P** intersect in a line **AB**. Prove that any cone whose vertex is **O** and whose base is the section of the ellipsoid by a plane through **AB** is right circular.

64. If the generators through a point **P** on the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

meet the principal elliptic section in **A** and **B**, and if the eccentric angle of **A** is three times the eccentric angle of **B**, prove that **P** lies on the curve of intersection of the hyperboloid and the cylinder

$$y^2(z^2+c^2)=4b^2z^2.$$

65. If **P** is an extremity of an equiconjugate diameter of the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$ and $2c^2=a^2+b^2$, the two generators through **P** of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ are at right angles.

66. **P** and **D** are the extremities of conjugate diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and the generators through **P** and **D** form a skew quadrilateral **PQDR**. If the angles **QPR**, **QDR** are 2θ and 2ϕ, prove that

$$\cot^2\theta+\cot^2\phi=(a^2+b^2)/c^2.$$

67. A generator of $x^2+y^2-2z^2=a^2$ through the point $(a\cos\alpha, a\sin\alpha, 0)$ intersects a generator through the point $(a\cos\beta, a\sin\beta, 0)$ at an angle θ. Prove that $3\cos^2\theta/2=2\cos^2(\alpha-\beta)/2$, and show that if A_1, A_2, A_3, A_4, A_5, A_6 are the vertices of a regular hexagon inscribed in the principal circular section, the generators of the one system through A_1, A_3, A_5 and the generators of the other system through A_2, A_4, A_6 are the edges of a cube whose volume is $2\sqrt{2}a^3$.

68. If $a^2>bc$ show that the points of intersection of perpendicular generators of the hyperboloid $\frac{x^2}{a^2}-\frac{2yz}{bc}=1$ lie on the real central circular sections.

69. The generators through **S**, any point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

meet the plane $lx+my+nz=1$ in the points **P** and **Q**. The other generators through **P** and **Q** intersect in **R**. Find the equation of the plane **PQR**, and prove that if **R** always lies on the plane $z=0$, **S** lies on the plane

$$z(a^2l^2+b^2m^2+c^2n^2-1)+2nc^2(lx+my-1)=0.$$

70. If a hyperboloid passes through six edges of a cube it must be formed by the revolution of a hyperbola of eccentricity $\sqrt{3}/2$ about a diagonal of the cube.

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